

ON THE REGULARITY ISSUES OF A CLASS OF DRIFT-DIFFUSION EQUATIONS WITH NONLOCAL DIFFUSION

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ABSTRACT. In this paper we address the regularity issues of drift-diffusion equation with nonlocal diffusion, where the diffusion operator is in the realm of stable-type Lévy operator and the velocity field is defined from the considered quantity by some zero-order pseudo-differential operators. Through using the method of nonlocal maximum principle in a unified way, we prove the global well-posedness result in some slightly supercritical cases, and show the eventual regularity result in the supercritical type cases. The time after which the solution is smoothly regular in the supercritical type cases can be evaluated appropriately, so that we can prove a type of global result recently obtained by [17] and also show the global regularity of vanishing viscosity solution at some logarithmically supercritical cases.

1. INTRODUCTION

In this article we consider the Cauchy problem of the following drift-diffusion equation with nonlocal diffusion

$$\partial_t \theta + u \cdot \nabla \theta + \mathcal{L} \theta = 0, \quad \theta|_{t=0}(x) = \theta_0(x), \quad (1.1)$$

where $x \in \mathbb{R}^d$ (or \mathbb{T}^d), $d \in \mathbb{N}^+$, $t \in \mathbb{R}^+$, θ is a scalar-valued quantity understood as density or temperature field, and the velocity field $u = \mathcal{P}(\theta)$ is a vector field of \mathbb{R}^d defined from θ by the zero-order pseudo-differential operator:

$$u(x) = \mathcal{P}(\theta)(x) = a \theta(x) + \text{p.v.} \int_{\mathbb{R}^d} S(y) \theta(x+y) dy, \quad (1.2)$$

with $a = (a_1, \dots, a_d) \in \mathbb{R}^d$, and $S(x) = \frac{\Psi(x/|x|)}{|x|^d} = \left(\frac{\Psi_1(x/|x|)}{|x|^d}, \dots, \frac{\Psi_d(x/|x|)}{|x|^d} \right) \in C(\mathbb{R}^d \setminus \{0\}; \mathbb{R}^d)$ composed of Calderón-Zygmund kernels ([36]). The nonlocal diffusion operator \mathcal{L} is given by

$$\mathcal{L} f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(x) - f(x+y)) K(y) dy, \quad (1.3)$$

where the radially symmetric kernel function $K(y) = K(|y|)$ defined on $\mathbb{R}^d \setminus \{0\}$ satisfies that there exist some $\alpha \in]0, 1]$, $\tilde{\alpha} > 0$ and $c_0 > 0$ (c_0 may be dependent on α and σ below), $c_1 \geq 1$ such that

$$c_1^{-1} \frac{m(|y|^{-1})}{|y|^d} \leq K(y) \leq c_1 \frac{m(|y|^{-1})}{|y|^d}, \quad \forall 0 < |y| \leq c_0, \quad \text{and} \quad (1.4)$$

$$0 \leq K(y) \leq \frac{c_1}{|y|^{d+\tilde{\alpha}}}, \quad \forall |y| \geq c_0, \quad (1.5)$$

with $m(y) = m(|y|)$ a radially symmetric function satisfying the following assumptions

- (i) $m(|y|)$ is smooth away from zero, non-decreasing, with $m(0) = 0$, $\lim_{|y| \rightarrow \infty} m(|y|) = \infty$;
- (ii) there exists $\sigma \in [0, \alpha[$ such that

$$(\alpha - \sigma) \frac{m(|y|)}{|y|} \leq m'(|y|) \leq \alpha \frac{m(|y|)}{|y|}, \quad \forall |y| \geq c_0^{-1}. \quad (1.6)$$

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In some cases, the condition (1.5) can be replaced by a more general condition

$$-\frac{c_1}{|y|^{d+\alpha}} \leq K(y) \leq \frac{c_1}{|y|^{d+\alpha}}, \quad \forall |y| \geq c_0. \quad (1.7)$$

Besides, we also consider the nonlocal operator \mathcal{L} defined by (1.3)-(1.6) with “ $c_0 = \infty$ ”, i.e., the kernel $K(y) = K(|y|)$ is given by

$$c_1^{-1} \frac{m(|y|^{-1})}{|y|^d} \leq K(y) \leq c_1 \frac{m(|y|^{-1})}{|y|^d}, \quad \forall y \in \mathbb{R}^d \setminus \{0\}, \quad (1.8)$$

with $c_1 \geq 1$ and $m(y) = m(|y|)$ satisfying (i) and

(ii)’ there exists a constant $\sigma \in [0, \alpha[$ such that

$$(\alpha - \sigma) \frac{m(|y|)}{|y|} \leq m'(|y|) \leq \alpha \frac{m(|y|)}{|y|}, \quad \forall |y| > 0. \quad (1.9)$$

The diffusion operator (1.3) defined above is in the realm of Lévy operator; indeed, according to (1.6) and Lemma 2.2 below, we deduce that for $\alpha \in]0, 1]$ and $\sigma \in [0, \alpha[$,

$$\frac{c_0^{\alpha-\sigma} m(c_0^{-1})}{|y|^{\alpha-\sigma}} \leq m(|y|^{-1}) \leq \frac{c_0^\alpha m(c_0^{-1})}{|y|^\alpha}, \quad \forall 0 < |y| \leq c_0, \quad (1.10)$$

which leads to

$$\frac{c_1^{-1} c_0^{\alpha-\sigma} m(c_0^{-1})}{|y|^{d+\alpha-\sigma}} \leq K(y) \leq \frac{c_1 c_0^\alpha m(c_0^{-1})}{|y|^{d+\alpha}}, \quad \forall 0 < |y| \leq c_0, \quad (1.11)$$

and we know that the operator given by (1.3) satisfying (1.11) and $\int_{\mathbb{R}^d} (\min\{1, |y|^2\}) K(y) dy \leq C$ corresponds to the infinitesimal generator of the stable-type Lévy process (cf. [8, 35]). By taking the Fourier transform on \mathcal{L} , we get

$$\widehat{\mathcal{L}f}(\zeta) = A(\zeta) \widehat{f}(\zeta), \quad \forall \zeta \in \mathbb{R}^d, \quad (1.12)$$

where the symbol $A(\zeta)$ is given by the following Lévy-Khinchin formula

$$A(\zeta) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\zeta \cdot y)) K(y) dy. \quad (1.13)$$

The diffusion operator \mathcal{L} defined by (1.3) under the kernel conditions (1.4)-(1.5) or (1.4), (1.7) contains a large class of multiplier operators $\mathcal{L} = m(D)$ such as

$$\mathcal{L} = |D|^\beta, \quad (\beta \in [\alpha - \sigma, \alpha]), \quad \text{and} \quad \mathcal{L} = \frac{|D|^\alpha}{(\log(\lambda + |D|))^\mu}, \quad (\alpha \in]0, 1], \mu > 0, \lambda \geq 1),$$

which we will explain in the subsection 2.1 below. Among them, an important case, which is also a particular case of \mathcal{L} under the kernel conditions (1.8)-(1.9), is the fractional Laplacian operator $|D|^\alpha := (-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in]0, 1]$, which has the following representation formula

$$|D|^\alpha f(x) = c_{d,\alpha} \text{p.v.} \int_{\mathbb{R}^d} \frac{f(x) - f(x+y)}{|y|^{d+\alpha}} dy, \quad (1.14)$$

with $c_{d,\alpha} > 0$ some absolute constant. The operator $\mathcal{L} = |D|^\alpha$ corresponds to the infinitesimal generator of the symmetric stable Lévy process, and recently has been intensely studied in many theoretical problems. For the drift-diffusion equation (1.1)-(1.2) with $\mathcal{L} = |D|^\alpha$, we conventionally call the cases $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$ as supercritical, critical and subcritical cases, respectively.

The drift-diffusion equation (1.1)-(1.2) has various physical background from the geophysics, fluid dynamics, dislocation theory and other fields. The typical examples are the surface quasi-geostrophic equation, the Burgers equation, the Córdoba-Córdoba-Fontelos equation and the incompressible porous media equation, and below we will specifically review some noticeable results related to these models. For other interesting models expressed as the equation (1.1)-(1.2), one can also refer to [3, 22, 29] etc.

The surface quasi-geostrophic (SQG) equation corresponds to the equation (1.1) with

$$d = 2 \quad \text{and} \quad u = \mathcal{R}^\perp \theta = (-\mathcal{R}_2, \mathcal{R}_1)\theta, \quad (1.15)$$

where $\mathcal{R}_i = \partial_i |D|^{-1}$ ($i = 1, 2$) is the usual Riesz transform. The inviscid model (i.e. $\mathcal{L} = 0$) arises from the geostrophic study of the highly rotating fluid (cf. [33]), and partially due to the formal analogue with 3D Euler/Navier-Stokes equations (cf. [9]) and its simple form, the SQG equation has received much attention. For the SQG equation with fractional operator $\mathcal{L} = |D|^\alpha$, the subcritical case (i.e. $\alpha \in]1, 2[$) has been known for a while that it is globally well-posed for suitably regular data (e.g. [34]); while for the subtle critical case (i.e. $\alpha = 1$), the issue of global regularity was independently settled by [28] and [4]. Kiselev et al in [28] developed an original method called the “nonlocal maximum principle”; and Caffarelli et al in [4] exploited the De Giorgi’s iteration method. For other quite different proofs resolving the critical problem, one can refer to [26] which uses the duality method, and [11, 10] which apply the “nonlinear maximum principle” method. However, the global regularity issue in the supercritical case remains to be an outstanding open problem. So far, for the SQG equation with supercritical diffusion (i.e. $\alpha \in]0, 1[$), we only know some partial results: the local well-posedness result for large data and global well-posedness result under some smallness condition (e.g. [7]), the conditional regularity criterion (e.g. [12]), and the eventual regularity of the global weak solution (cf., [18, 25, 17]). More precisely, for the eventual regularity issue, which means the global weak solution is smoothly regular after some finite time, the progress was first made by Dabkowski [18] by adapting the method of [26] and later achieved by Kiselev [25] by using the nonlocal maximum principle method, and one refer to [17] for a third proof by applying the method of [10]. We also notice that Coti Zelati and Vicol in [17] also proved a somewhat global result that for $\theta_0 \in H^2$ with $\|\theta_0\|_{L^2}^{\alpha/2} \|\theta_0\|_{\dot{H}^2}^{1-\alpha/2} \leq R$, the supercritical SQG equation has a unique global solution as long as α depending on R sufficiently close to 1. For the SQG equation with general diffusion operator \mathcal{L} , Dabkowski et al in [19] considered the *slightly supercritical case*, where the operator \mathcal{L} defined by (1.3) and (1.8) satisfies (1.22) below, and they obtained the global well-posedness of smooth solution by applying the method of nonlocal maximum principle. They also showed the global result for the multiplier operator $\mathcal{L} = m(D)$ under some suitable assumptions on $m(\zeta) = m(|\zeta|)$.

The Burgers equation is just the equation (1.1) with

$$d = 1, \quad \text{and} \quad u = \theta, \quad (1.16)$$

which was introduced and studied by Burgers in 1940s as a 1D equation modeling the nonlinearity of 3D Euler/Navier-Stokes equations. It is known that the inviscid Burgers equation with some smooth data forms the shock singularity at finite time. For the Burgers equation with fractional diffusion, the subcritical and critical cases can be treated as the corresponding cases of SQG equation to obtain the global results; while for the supercritical case, Kiselev et al in [27] proved that the shock singularity similar to the inviscid case occurs in the supercritical case (see also [1]). For the Burgers equation with a general \mathcal{L} defined by (1.3) and (1.8), the authors in [19] proved that under (1.22) below and other mild conditions on m , the equation is globally well-posed for smooth data; whereas under $\lim_{\nu \rightarrow 0+} \int_\nu^1 m(r^{-1}) dr < \infty$, finite time blowup will happen for some smooth data.

The Córdoba-Córdoba-Fontelos (CCF) equation corresponds to the equation (1.1) with

$$d = 1, \quad \text{and} \quad u = H\theta, \quad (1.17)$$

and H is the usual 1D Hilbert transform. Córdoba et al in [16] introduced this model as a 1D simple equation of 3D Euler/Navier-Stokes equations which has the nonlocal velocity; and they proved there exists smooth data so that the inviscid CCF equation forms singularity at finite time. For the CCF equation with fractional diffusion, Dong in [21] considered the subcritical and critical cases, and showed the global results, while in the supercritical case with $\alpha \in]0, 1/2[$, Li et al in [30] showed there is an occurrence of finite-time blowup similar to the inviscid case. Up to now, the problem concerning

the global regularity of solution for the supercritical CCF equation with $\alpha \in [1/2, 1[$ is still open. We mention that Do in [20] solved the eventual regularity of the global weak solution for all the supercritical case $\alpha \in]0, 1[$ by applying the method of [25], and also proved the global well-posedness result of the CCF equation at some slightly supercritical cases.

The incompressible porous media equation is the equation (1.1) with the following velocity field

$$u = \nabla p + \theta e_d, \quad \operatorname{div} u = 0, \quad (1.18)$$

where p is a scalar quantity and e_d is the last canonical vector of \mathbb{R}^d . By a direct computation, we can show that the velocity u can be exactly expressed as (1.2), e.g., for $d = 2$ (cf. [15]),

$$a = \left(0, -\frac{1}{2}\right), \quad S(x) = \frac{1}{2\pi} \left(\frac{2x_1x_2}{|x|^4}, \frac{x_2^2 - x_1^2}{|x|^4}\right),$$

and for $d = 3$ (cf. [5]),

$$a = \left(0, 0, -\frac{2}{3}\right), \quad S(x) = \frac{1}{4\pi} \left(\frac{3x_1x_3}{|x|^5}, \frac{3x_2x_3}{|x|^5}, \frac{2x_3^2 - x_1^2 - x_2^2}{|x|^5}\right).$$

In [5, 15], Córdoba et al, among other issues, proved the global well-posedness result for the equation in the subcritical and critical cases. Similarly as the SQG equation, the issue of global regularity in the supercritical case remains unsolved.

In this paper we focus on the drift-diffusion equation (1.1)-(1.2) with general \mathcal{L} defined by (1.3), and we mainly are concerned with the following cases

$$\text{Case (I): } K(|y|) \text{ satisfies (1.4)-(1.5), } m(|y|) \text{ satisfies (i)-(ii);} \quad (1.19)$$

$$\text{Case (II): } K(|y|) \text{ satisfies (1.4), (1.7), } m(|y|) \text{ satisfies (i)-(ii),}$$

$$A(\zeta) \geq 0, \forall \zeta \in \mathbb{R}^d, \text{ and } \operatorname{div} u = 0; \quad (1.20)$$

$$\text{Case (III): } K(|y|) \text{ satisfies (1.8), } m(|y|) \text{ satisfies (i), (ii)'.} \quad (1.21)$$

By applying the method of nonlocal maximum principle in a unified way, we show the global well-posedness result for the slightly supercritical drift-diffusion equation (1.1)-(1.2) at either Case (I) or Case (II), and the eventual regularity of global weak solution for the supercritical type equation (1.1)-(1.2) at Case (III). Compared with the eventual result obtained in [25] for the supercritical SQG equation, we have an explicit control on the eventual regularity time (i.e., the time after which the solution is regular) which is small enough as $\sigma \rightarrow 0$, $\alpha = 1$ or under the condition (1.22). By using this point, we prove a type of global result recently obtained by Coti Zelati and Vicol in [17], and also get the global regularity of vanishing viscosity solution for the equation (1.1)-(1.2) at some logarithmically supercritical cases.

Precisely, our first result is the global well-posedness for the the slightly supercritical equation (1.1)-(1.2), partially generalizing the result of [19] on the slightly supercritical SQG and Burgers equations.

Theorem 1.1. *Assume that $\theta_0 \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2} + 1$, and either Case (I) or Case (II) is considered with $\alpha \in]0, 1]$, $\sigma \in [0, 1[$ and some constant $c_0 = c_{\alpha, \sigma} > 0$. In addition, suppose the radial function m appearing in K satisfies*

$$\lim_{\nu \rightarrow 0+} \int_{\nu}^{c_{\alpha, \sigma}} m(\xi^{-1}) d\xi = \infty, \quad (1.22)$$

then the associated drift-diffusion equation (1.1)-(1.2) generates a uniquely global smooth solution θ such that

$$\theta \in C([0, \infty[; H^s(\mathbb{R}^d)) \cap C^\infty([0, \infty[\times \mathbb{R}^d). \quad (1.23)$$

Remark 1.2. *For $m(|y|) = \frac{|y|}{(\log(\lambda + |y|))^\mu}$ with $\mu \in [0, 1]$, $\lambda \geq 0$, we see that (1.6) with $\alpha = 1$ and $\sigma \in]0, 1[$ is satisfied for all $|y| \geq c_0 = e^{-\frac{\mu}{\sigma}}$, and also (1.22) holds true, thus according to Theorem 1.1,*

we can prove the global well-posedness of the smooth solution for either Case (I) or Case (II) equipped with such m and c_0 . Similar results also hold for $m(|y|) = \frac{|y|}{\log(\lambda_1 + |y|)(\log \log(\lambda_2 + |y|))^\mu}$ with $\mu \in [0, 1]$, $\lambda_1, \lambda_2 \geq 0$, and so on.

The following crucial result is concerned with the uniform-in- ϵ improvement from L^∞ -solution to Hölder continuous solution after some finite time for the ϵ -regularized equation (1.24).

Proposition 1.3. *Assume that Case (III) is considered with $\alpha \in]0, 1[$, $\sigma \in [0, 1[$, and $\theta \in C([0, \infty[; H^s(\mathbb{R}^d))$, $s > 1 + \frac{d}{2}$ is a smooth solution for the following regularized drift-diffusion equation*

$$\partial_t \theta + u \cdot \nabla \theta + \mathcal{L} \theta - \epsilon \Delta \theta = 0, \quad \theta|_{t=0} = \theta_0, \quad (1.24)$$

where $\epsilon > 0$, $\theta_0 \in L^\infty$, u is given by (1.2). Then there exists a time $t_1 > 0$ independent of ϵ such that for every $\beta \in]1 - \alpha + \sigma, 1[$,

$$\sup_{t \in [t_1, \infty[} \|\theta(t)\|_{\dot{C}^\beta(\mathbb{R}^d)} \leq C(\|\theta_0\|_{L^\infty}, d, \alpha, \beta, \sigma), \quad (1.25)$$

with C a constant independent of ϵ . Besides, if $\alpha \in]0, 1[$ and $\sigma = 0$ in the condition (1.9), we have the explicit estimates on t_1 and $\sup_{t \in [t_1, \infty[} \|\theta(t)\|_{\dot{C}^\beta}$ as (5.12)-(5.13) below.

A direct consequence of Proposition 1.3 is the eventual regularity of the vanishing viscosity weak solution for the drift-diffusion equation (1.1)-(1.2).

Theorem 1.4. *Assume that $\theta_0 \in L^2(\mathbb{R}^d)$, $\operatorname{div} u = 0$ and Case (III) is considered with $\alpha \in]0, 1[$, $\sigma \in [0, 1[$. Then for every $T > 0$ large enough, the drift-diffusion equation (1.1)-(1.2) admits a weak solution $\theta(x, t)$ on $[0, T]$, which is $C_x^{1, \nu}$ -regular with some $\nu > 0$ for every $t \in]t_0 + t_1, T]$, where $t_0 > 0$ can be chosen arbitrarily small and $t_1 > 0$ is a number depending only on α, σ, d, t_0 and $\|\theta_0\|_{L^2}$.*

Besides, if $\alpha \in]0, 1[$ and $\sigma = 0$ in the condition (1.9), i.e., $m(y) \equiv C_0 |y|^\alpha$ ($\alpha \in]0, 1[$, $\forall y \neq 0$), we can choose $T = \infty$, and we explicitly have

$$t_1 \leq \frac{C}{\alpha} \left(C 2^{d/\alpha} t_0^{-1} \right)^{\frac{d}{2(1-\alpha)}} \left(\frac{C(1-\alpha)}{\alpha^5} \right)^{\frac{\alpha}{1-\alpha}} \|\theta_0\|_{L^2}^{\frac{\alpha}{1-\alpha}}, \quad (1.26)$$

with $C > 0$ some constant depending only on d .

Motivated by [17], and as an another consequence of Proposition 1.3, we can prove the following global result.

Theorem 1.5. *Assume that either Case (I) or Case (II) is considered for $\alpha = 1$ and $\sigma \in [0, 1[$ with some constant $c_0 > 0$ (independent of σ). Let $\theta_0 \in H^s(\mathbb{R}^d)$, $s > 1 + \frac{d}{2}$ be such that $\|\theta_0\|_{H^s(\mathbb{R}^d)} \leq R$ with some $R > 0$. Then there exists a constant $\sigma_1 = \sigma_1(R, d) > 0$ such that for every $\sigma \leq \sigma_1$, the associated drift-diffusion equation (1.1)-(1.2) has a unique global solution $\theta(x, t)$ satisfying (1.23).*

Remark 1.6. *For Case (I) and Case (II) satisfying that $m(y) = \frac{|y|^{1-\delta}}{(\log(\lambda_\delta + |y|))^\mu}$ for all $|y| \geq c_0^{-1}$ with $\delta \in [0, 1[$, $c_0 \in]0, 1[$ and $\lambda_\delta, \mu > 0$, we see that (1.6) holds true for $\alpha = 1$ and $\sigma = \delta + \frac{\mu}{\log(\lambda_\delta + c_0^{-1})}$, thus by choosing δ small enough and λ_δ large enough so that $\sigma \leq \sigma_1$, Theorem 1.5 can be applied to obtain a type of global result.*

As a counterpart of Theorem 1.1, and also as a consequence of Proposition 1.3 and Theorem 1.5, we prove the global regularity of vanishing viscosity solution for some logarithmically supercritical equations (1.1)-(1.2).

Theorem 1.7. *Assume that either Case (I) or Case (II) is considered for $\alpha = 1$ and $\sigma \in [0, 1[$ with some constant $c_0 = c_\sigma > 0$. Additionally suppose that there exist $\mu \in [0, 1]$ and $c_2 \geq 1$ such that*

$$\frac{1}{c_2} \frac{|y|}{(\log |y|)^\mu} \leq m(|y|) \leq c_2 |y|, \quad \forall |y| \geq c_2. \quad (1.27)$$

Let $\theta_0 \in C_0(\mathbb{R}^d)$ in the Case (I), or $\theta_0 \in L^2 \cap L^\infty(\mathbb{R}^d)$ in the Case (II). Then for any $t_* > 0$ small, the corresponding vanishing viscosity solution θ of the drift-diffusion equation (1.1)-(1.2) belongs to $C^\infty([t_*, \infty[\times \mathbb{R}^d)$.

Remark 1.8. Since $m(|y|) = \frac{|y|}{(\log(\lambda + |y|))^\mu}$ with $\mu \in [0, 1]$, $\lambda \geq 0$ satisfies (1.6) with $\alpha = 1$, $\sigma \in]0, 1[$, $c_0 = e^{-\frac{\mu}{\sigma}}$, and also satisfies (1.27) with $c_2 = 2$, thus Theorem 1.7 can be applied to the equation (1.1)-(1.2) under either Case (I) or Case (II) with these m and c_0 . Recalling that the improvement from L^∞ to Hölder regularity is a crucial step in proving the global regularity of weak solution for the critical SQG equation (i.e. $\mathcal{L} = |D|$) by Caffarelli-Vasseur [4] and also Kiselev-Nazarov [26], we here as a nontrivial generalization achieve such an improvement for vanishing viscosity solution of the drift-diffusion equation (1.1)-(1.2) at some logarithmically supercritical cases, and we even remove the divergence-free assumption of the velocity field at Case (I).

Remark 1.9. As already observed by several authors in the literature, the SQG equation (in the inviscid or supercritical case) may be the simplest physical PDE model that the issue of global regularity still remains open. The results in this paper improve and extend some noticeable results of SQG equation in [25, 19, 17] to the drift-diffusion equation (1.1) with a general velocity field given by (1.2). But since we only use the representation formula (1.2) (and the divergence-free condition of u in some cases) and do not use the exclusive properties of the Riesz transform (cf. [36, Chapter III]), it seems that so far the special structure of the velocity field (1.15) do not play an indispensable role on deriving the already obtained main results in the regularity issues of SQG equation.

The main method in proving the above results is the nonlocal maximum principle (cf. [28, 25]), whose basic idea is to show the evolution preserves some appropriate modulus of continuity (see Section 3 below).

For Theorem 1.1, the local well-posedness result is stated and proved in the appendix section 7, then we introduce a MOC $\omega(\xi)$ defined by (4.1) and prove that the evolution of the considered equation (1.1)-(1.2) obeys this MOC, which implies the needed Hölder regularity of the solution. Compared with [19], which adapted the same method for the slightly supercritical SQG equation, the MOC (4.1) has a much simpler form, and we use a different way to estimate the contribution (3.16) so that we can avoid the difficulty encountered in considering the general u defined by (1.2) (since the treating in [19] uses the special structure of $u = \mathcal{R}^\perp \theta$ and do not extend).

Proposition 1.3 concerns the uniform-in- ϵ improvement of the eventual Hölder regularity from the L^∞ -weak solution for the solution of ϵ -regularized equation 1.24, which indeed plays a core role in the proof of Theorems 1.4, 1.5 and 1.7. For the proof of Proposition 1.3, a new ingredient is the MOC $\omega(\xi, \xi_0)$ given by (5.1)-(5.2), which is derived from suitably modifying the MOC (4.1), and by virtue of a careful analysis according to the values of ξ and ξ_0 , we can show that the solution $\theta(x, t)$ of the regularized equation (1.24) uniformly obeys some appropriate MOC $\omega(\xi, \xi_0(t))$, which further implies the desired uniform-in- ϵ Hölder regularity estimate after some time. We stress that there is no factor like $1 - \alpha + \sigma$ or $1 - \alpha$ in the conditions of κ, γ, ρ (see (5.56)) appearing in $\omega(\xi, \xi_0)$, so that we can estimate the eventual regularity time t_1 as (1.26) in the case $\alpha \in]0, 1[$, $\sigma = 0$, which has the property that $t_1 \rightarrow 0$ as $\alpha \rightarrow 1$ for the fixed data θ_0 (note that such a property on the eventual regularity time for the supercritical SQG equation was not achieved in [25]).

For the proof of Theorem 1.4, we first prove the global existence of a vanishing viscosity solution satisfying the L^2 -energy estimate, then by using De Giorgi's method we show the crucial L_x^∞ -improvement for all $t \geq t_0$ with any $t_0 > 0$, and then Proposition 1.3 ensures the eventual Hölder regularity of this weak solution for every $t \geq t_0 + t_1$ with some $t_1 > 0$, which in combination with the regularity criterion Lemma 2.5 further leads to the desired eventual regularity result.

For Theorem 1.5, we prove the expected global result by combining the local well-posedness result in Theorem 7.1 and the eventual regularity result in Proposition 1.3, especially using the key point

that the eventual regularity time t_1 is well estimated. Note that in the considered case it suffices to justify the criterion (6.22) for small ξ and $\xi_0(t)$, so that we can treat the more general diffusion operator \mathcal{L} than in Proposition 1.3.

For Theorem 1.7, by applying Proposition 1.3 and Theorem 1.5, we see that under the condition (1.27), the eventual regularity time t_1 can be arbitrarily small, and thus by appropriately choosing the coefficients in the MOC $\omega(\xi, \xi_0)$ and $\xi_0 = \xi_0(t)$, we can show the desired global regularity result.

The outline of the paper is as follows. In Section 2, we introduce a class of multiplier operators as examples of the diffusion operator \mathcal{L} , and we present some useful auxiliary lemmas. In the section 3, some basic and useful results related to the modulus of continuity are collected. In Section 4 we prove the desired Hölder regularity of the solution, which further concludes Theorem 1.5. In the section 5, we give the detailed proof of Proposition 1.3, which is concerned with the crucial eventual Hölder regularity issue. The proof of Theorems 1.4, 1.5 and 1.7 are respectively placed in the subsections of Section 6. At last, the appendix section sketches the proof of the local well-posedness result for the considered drift-diffusion equation.

Notations. Throughout this paper, C stands for a constant which may be different from line to line. The notion $X \lesssim Y$ means that $X \leq CY$. Denote $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions. We use \widehat{f} and \check{g} to denote the Fourier transform and the inverse Fourier transform of a tempered distribution, that is, $\widehat{f}(\zeta) = \int_{\mathbb{R}^d} e^{ix \cdot \zeta} f(x) dx$ and $\check{g}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \zeta} g(\zeta) d\zeta$.

2. PRELIMINARY

In this section, we introduce a class of multiplier operators as examples of the operator \mathcal{L} , and also compile several useful auxiliary lemmas.

2.1. Multiplier operators as examples of \mathcal{L} . In addition to the conditions (i)-(ii) stated in the introduction, we assume that the function $m(\zeta) = m(|\zeta|)$ also may satisfy the following assumptions: (iii) m is of the Mikhlin-Hörmander type, i.e. there is some constant $c_3 \geq 1$ so that

$$|\partial_\zeta^k m(\zeta)| \leq c_3 |\zeta|^{-k} m(\zeta), \quad \forall \zeta \neq 0, \quad (2.1)$$

for all $k \in \mathbb{N}$ and $k \leq k_0$, with k_0 a positive constant depending only on d .

(iv) m satisfies that

$$(-\Delta)^d m(\zeta) \geq c_4 |\zeta|^{-2d} m(\zeta), \quad \forall |\zeta| \text{ large enough}, \quad (2.2)$$

with some $c_4 > 0$.

(v) m satisfies that

$$(-1)^{k-1} m^{(k)}(|\zeta|) \geq 0, \quad \forall |\zeta| > 0, k \in \{1, 2, \dots, d\}. \quad (2.3)$$

Note that there do exist a large class of nontrivial examples satisfying all the needing conditions; in fact, as shown by [23, Proposition 3.6], the functions $m(\zeta) = \frac{|\zeta|^\alpha}{(\log(\lambda + |\zeta|))^\beta}$ with $\lambda \geq e^{\frac{3+2\beta}{\alpha}}$, $\alpha \in]0, 1]$, $\beta \geq 0$ and $d = 1, 2, 3$ satisfy (2.3), and they also satisfy the conditions (i)-(iv) by a direct computation.

The following lemma relates the multiplier operator with the conditions of K in the introduction.

Lemma 2.1. *Suppose that $m(\zeta) = m(|\zeta|)$ is a radial function satisfies the conditions (i)-(iv) stated above. Then the multiplier operator $m(D)$ has the following representation formula*

$$m(D)\theta(x) = \left(m(\zeta) \widehat{\theta}(\zeta) \right)^\vee(x) = \text{p.v.} \int_{\mathbb{R}^d} K(y) (\theta(x) - \theta(x+y)) dy, \quad (2.4)$$

where the radial kernel K satisfies

$$|K(y)| \leq C |y|^{-d} m(|y|^{-1}), \quad \forall |y| > 0, \quad (2.5)$$

and

$$K(y) \geq c_5 |y|^{-d} m(|y|^{-1}), \quad \forall 0 < |y| \leq c_0, \quad (2.6)$$

with two generic constants $c_0, c_5 > 0$. Besides, if $m(\zeta) = m(|\zeta|)$ additionally satisfies the condition (v), then the kernel function K in (2.4) also satisfies

$$K(y) \geq 0, \quad \forall |y| > 0. \quad (2.7)$$

Notice that the properties (2.5)-(2.6) just correspond to the conditions (1.4), (1.7), and the properties (2.5)-(2.7) correspond to the conditions (1.4)-(1.5).

Proof of Lemma 2.1. The properties (2.5)-(2.6) were proved in Lemmas 5.1 - 5.2 of [19]. We only prove (2.7). By arguing as Proposition 3.6 and Lemma 3.8 of [23], we can show that, thanks to (v), the kernel function $G_t(x)$ associated with the operator $e^{-t\mathcal{L}}$ satisfies

$$G_t(x) \geq 0, \quad \text{and} \quad \int_{\mathbb{R}^d} G_t(x) dx = \widehat{G_t(\cdot)}|_{\zeta=0} = 1.$$

In light of the semigroup representation formula of the operator \mathcal{L} ,

$$\mathcal{L}f(x) = \lim_{t \rightarrow 0+} \frac{f(x) - e^{-t\mathcal{L}}f(x)}{t} = \lim_{t \rightarrow 0+} \int_{\mathbb{R}^d} \frac{G_t(y)}{t} (f(x) - f(x+y)) dy,$$

we see that $K(y) = \lim_{t \rightarrow 0} \frac{G_t(y)}{t} \geq 0$ for all $|y| > 0$. \square

2.2. Auxiliary lemmas. First we prove a lemma on the property of the function m satisfying (1.6), which will be repeatedly used in the sequel.

Lemma 2.2. *Let $m(y) = m(|y|)$ be the radial function satisfying the condition (1.6) for some $\alpha \in]0, 1[$ and $\sigma \in [0, \alpha[$, then*

$$\text{the mapping } |y| \mapsto |y|^{\beta_1} m(|y|^{-1}), \quad \beta_1 \geq \alpha \text{ is non-decreasing,} \quad (2.8)$$

and

$$\text{the mapping } |y| \mapsto |y|^{\beta_2} m(|y|^{-1}), \quad \beta_2 \leq \alpha - \sigma \text{ is non-increasing.} \quad (2.9)$$

Proof of Lemma 2.2. Let $f_i(r) = r^{\beta_i} m(r^{-1})$ for $i = 1, 2$ and $r > 0$, then by direct computation,

$$f'_1(r) = r^{\beta_1-1} (\beta_1 m(r^{-1}) - r^{-1} m'(r^{-1})) \geq (\beta_1 - \alpha) r^{\beta_1-1} m(r^{-1}) \geq 0,$$

which yields (2.8), and similarly,

$$f'_2(r) = r^{\beta_2-1} (\beta_2 m(r^{-1}) - r^{-1} m'(r^{-1})) \leq (\beta_2 - (\alpha - \sigma)) r^{\beta_2-1} m(r^{-1}) \leq 0,$$

which yields (2.9). \square

The next lemma concerns the pointwise lower bound estimate of the symbol of the operator \mathcal{L} .

Lemma 2.3. *Let \mathcal{L} be defined by (1.3) with $K(|y|)$ satisfying (1.4)-(1.5) and $m(|y|)$ satisfying (i)-(ii), then the associated symbol $A(\zeta)$ given by (1.13) satisfies that*

$$A(\zeta) \geq C^{-1} |\zeta|^{\alpha-\sigma} - C, \quad \forall \zeta \in \mathbb{R}^d, \quad (2.10)$$

where $\alpha \in]0, 1[$, $\sigma \in [0, \alpha[$ and C is a positive constant depending only on d , α and σ . Besides, if $K(|y|)$ satisfies (1.4), (1.7) with $m(|y|)$ satisfying (i)-(ii), we can also get (2.10) with a different constant C . In particular, if $K(|y|)$ satisfies (1.8) with $m(y) = |y|^\alpha$ ($\alpha \in]0, 1[$), $\forall y \neq 0$, then we also get

$$A(\zeta) \geq C^{-1} |\zeta|^\alpha, \quad \forall \zeta \in \mathbb{R}^d, \quad (2.11)$$

with C a positive constant depending only on d and α .

Note that if $m(y) \equiv |y|^\alpha$, then we can get (2.10) with $\sigma = 0$ for the associated operator \mathcal{L} , and this special result in fact has appeared in the literatures, e.g. [6, Lemma 2.2].

Proof of Lemma 2.3. Recalling that for every $\alpha \in]0, 2[$ we have (cf. Eq. (3.219) of [24])

$$|\zeta|^\alpha = c_{d,\alpha} \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(y \cdot \zeta)) \frac{1}{|y|^{d+\alpha}} dy, \quad \forall \zeta \in \mathbb{R}^d \quad (2.12)$$

and by virtue of the lower bound of K in (1.4)-(1.5) and the fact $|y|^{\alpha-\sigma} m(|y|^{-1}) \geq c_0^{\alpha-\sigma} m(c_0^{-1})$ for all $0 < |y| \leq c_0$, we get that

$$\begin{aligned} A(\zeta) &\geq c_1^{-1} \int_{0 < |y| \leq c_0} (1 - \cos(y \cdot \zeta)) \frac{m(|y|^{-1})}{|y|^d} dy \\ &\geq c_1^{-1} c_0^{\alpha-\sigma} m(c_0^{-1}) \int_{0 < |y| \leq c_0} (1 - \cos(y \cdot \zeta)) \frac{1}{|y|^{d+(\alpha-\sigma)}} dy \\ &\geq c_1^{-1} c_0^{\alpha-\sigma} m(c_0^{-1}) \left(c_{d,\alpha}^{-1} |\zeta|^{\alpha-\sigma} - \int_{|y| \geq c_0} \frac{1}{|y|^{d+\alpha-\sigma}} dy \right) \\ &\geq C^{-1} |\zeta|^{\alpha-\sigma} - C. \end{aligned}$$

If K satisfies (1.4) and (1.7), we similarly obtain

$$\begin{aligned} A(\zeta) &\geq c_1^{-1} \int_{0 < |y| \leq c_0} (1 - \cos(y \cdot \zeta)) \frac{m(|y|^{-1})}{|y|^d} dy - c_1 \int_{|y| \geq c_0} (1 - \cos(y \cdot \zeta)) |K(y)| dy \\ &\geq c_1^{-1} c_0^{\alpha-\sigma} m(c_0^{-1}) \int_{0 < |y| \leq c_0} (1 - \cos(y \cdot \zeta)) \frac{1}{|y|^{d+\alpha-\sigma}} dy - c_1 c_0^{\alpha-\sigma} m(c_0^{-1}) \int_{|y| \geq c_0} \frac{1}{|y|^{d+\bar{\alpha}}} dy \\ &\geq C^{-1} |\zeta|^{\alpha-\sigma} - C. \end{aligned}$$

If $K(|y|)$ satisfies (1.8) with $m(y) = |y|^\alpha$ ($\alpha \in]0, 1]$), $\forall y \neq 0$, from (2.12) we see that $A(\zeta) \geq c_1^{-1} c_{d,\alpha}^{-1} |\zeta|^\alpha$, which corresponds to (2.11). \square

The following lemma is about L^∞ -estimate of smooth solution for the equation (1.1)-(1.2).

Lemma 2.4. *Let $\theta(x, t) \in C([0, T^*]; H^s(\mathbb{R}^d))$, $s > 1 + \frac{d}{2}$ be a smooth solution to the drift-diffusion equation (1.1)-(1.2). If Case (I) (i.e. (1.19)) is supposed, then we have*

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}, \quad \text{for all } t \in [0, T^*]. \quad (2.13)$$

Besides, if Case (II) (i.e. (1.20)) is assumed, we get

$$\|\theta(t)\|_{L^\infty} \leq C(\|\theta_0\|_{L^2 \cap L^\infty}, \alpha, \sigma, d), \quad \text{for all } t \in [0, T^*]. \quad (2.14)$$

Proof of Lemma 2.4. Due to that the kernel K is nonnegative on $\mathbb{R}^d \setminus \{0\}$, the proof of (2.13) is classical (cf. [14, Theorem 4.1] for $\mathcal{L} = |D|^\alpha$), and we here omit the details.

Next we prove (2.14). Thanks to that $\operatorname{div} u = 0$ and $A(\zeta) \geq 0$, by the L^2 -energy estimate (cf. (6.4) below), we get $\|\theta(t)\|_{L_x^2} \leq \|\theta_0\|_{L^2}$ for all $t \in [0, T^*]$. Now for every $t \in]0, T^*]$, assume that $x_t \in \mathbb{R}^d$ is some point satisfying $\theta(x_t, t) = \|\theta(t)\|_{L_x^\infty} =: M(t)$. According to

$$\left| \left\{ y \in \mathbb{R}^d : |\theta(x_t + y)| \geq \frac{M(t)}{2} \right\} \right| \leq \left(\frac{2\|\theta(t)\|_{L^2}}{M(t)} \right)^2 \leq \frac{4\|\theta_0\|_{L^2}^2}{M(t)^2},$$

and denoting by $r_t := \frac{4^{1/d}}{|B_1(0)|^{1/d}} \frac{\|\theta_0\|_{L^2}^{2/d}}{M(t)^{2/d}}$, we may set $M(t)$ large enough so that $r_t \leq \frac{c_0}{2}$. Taking advantage of (1.4), (1.7) and Lemma 2.2, we find (by arguing as [26, Lemma 4.1]),

$$\begin{aligned}
(\mathcal{L}\theta)(x_t, t) &\geq c_1^{-1} \int_{0 < |y| \leq c_0} (\theta(x_t, t) - \theta(x_t + y, t)) \frac{m(|y|^{-1})}{|y|^d} dy - 2c_1 M(t) \int_{|y| \geq c_0} \frac{1}{|y|^{d+\tilde{\alpha}}} dy \\
&\geq c_1^{-1} \frac{M(t)}{2} \int_{r_t \leq |y| \leq c_0} \frac{m(|y|^{-1})}{|y|^d} dy - 2c_1 M(t) \int_{|y| \geq c_0} \frac{1}{|y|^{d+\tilde{\alpha}}} dy \\
&\geq c_1^{-1} c_0^{\alpha-\sigma} m(c_0^{-1}) \frac{M(t)}{2} \int_{r_t \leq |y| \leq c_0} \frac{1}{|y|^{d+\alpha-\sigma}} dy - 2c_1 M(t) \int_{|y| \geq c_0} \frac{1}{|y|^{d+\tilde{\alpha}}} dy \\
&\geq c_1^{-1} c_0^{\alpha-\sigma} m(c_0^{-1}) \frac{M(t)}{2} \frac{|B_1(0)|}{\alpha - \sigma} \frac{1}{2r_t^{\alpha-\sigma}} - 2c_1 M(t) \frac{|B_1(0)|}{\tilde{\alpha}} \\
&= \frac{C_{\alpha, \sigma, d}}{\|\theta_0\|_{L^2}^{2(\alpha-\sigma)/d}} M(t)^{1+\frac{2(\alpha-\sigma)}{d}} - C_{\tilde{\alpha}, d} M(t).
\end{aligned}$$

Hence we see that

$$\frac{d}{dt} M(t) \leq -C_{\alpha, \sigma, d} \|\theta_0\|_{L^2}^{-\frac{2(\alpha-\sigma)}{d}} M(t)^{1+\frac{2(\alpha-\sigma)}{d}} + C_{\tilde{\alpha}, d} M(t),$$

and for $M(t)$ larger than the quantity $\|\theta_0\|_{L^2} \left(\frac{C_{\tilde{\alpha}, d}}{C_{\alpha, \sigma, d}} \right)^{\frac{d}{2(\alpha-\sigma)}}$, we have $\frac{d}{dt} M(t) \leq 0$, which readily implies that $M(t) \leq \max \left\{ \|\theta_0\|_{L^\infty}, \left(\frac{C_{\tilde{\alpha}, d}}{C_{\alpha, \sigma, d}} \right)^{\frac{d}{2(\alpha-\sigma)}} \|\theta_0\|_{L^2} \right\}$ and concludes the proof. \square

Finally, we state the following key regularity criterion for the drift-diffusion equation (1.1).

Lemma 2.5. (1) Suppose that Case (I) is considered, $\theta_0 \in C_0(\mathbb{R}^d)$, and for $T > 0$ any given, the drift u satisfy

$$u \in L^\infty([0, T]; C^\delta(\mathbb{R}^d)), \quad \text{for every } \delta \in]1 - \alpha + \sigma, 1[, \quad (2.15)$$

then the drift-diffusion equation (1.1) admits a (classical) solution $\theta \in L^\infty([0, T]; C_0(\mathbb{R}^d)) \cap L^\infty(]0, T], C^{1, \gamma}(\mathbb{R}^d))$ for any $\gamma \in]0, \delta + \alpha - \sigma - 1[$ which is derived by passing $\epsilon \rightarrow 0$ of the regularized solution θ^ϵ solving the following approximate equation

$$\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon + \mathcal{L} \theta^\epsilon = 0, \quad u^\epsilon = \phi_\epsilon * u, \quad \theta^\epsilon|_{t=0} = \theta_0 1_{B_{1/\epsilon}}(x),$$

with $\phi_\epsilon(x) = \epsilon^{-d} \phi(\frac{x}{\epsilon})$ and ϕ the standard mollifier. Moreover, if the drift field u is given by (1.2), we have $\theta \in C^\infty(]0, T] \times \mathbb{R}^d)$.

(2) Assume that Case (II) is considered, $\theta_0 \in L^p(\mathbb{R}^d)$ for some $p \in [2, \infty)$, and the drift u satisfies (2.15) for $T > 0$ any given. Then the drift-diffusion equation (1.1) admits a unique weak solution (in the distributional sense) $\theta \in L^\infty([0, T]; L^p(\mathbb{R}^d))$ which satisfies $\theta \in L^\infty(]0, T], C^{1, \gamma}(\mathbb{R}^d))$ with any $\gamma \in]0, \delta + \alpha - \sigma - 1[$. Moreover, if the drift field u is given by (1.2), we have $\theta \in C^\infty(]0, T] \times \mathbb{R}^d)$.

For the proof of Lemma 2.5, one can refer to [38] for the detailed proof for the drift-diffusion equation (1.1) with more general diffusion operator \mathcal{L} .

3. MODULUS OF CONTINUITY

In this section we gather some results related to the modulus of continuity, which play an important role on the method of nonlocal maximum principle.

First is the definition of the modulus of continuity.

Definition 3.1. A function $\omega :]0, \infty[\rightarrow]0, \infty[$ is called a modulus of continuity (abbr. MOC) if ω is continuous on $]0, \infty[$, nondecreasing, concave, and piecewise C^2 with one-sided derivatives defined at every point in $]0, \infty[$. We say a function $f : \mathbb{R}^d \rightarrow \mathbb{R}^l$ obeys the modulus of continuity ω if $|f(x) - f(y)| < \omega(|x - y|)$ for every $x \neq y \in \mathbb{R}^d$.

Then we recall the general criterion of the nonlocal maximum principle for the whole-space drift-diffusion equation (for the proof see [32, Proposition 3.2] or [25, Theorem 2.2]).

Proposition 3.2. Let $\theta(x, t) \in C([0, \infty[; H^s(\mathbb{R}^d))$, $s > \frac{d}{2} + 1$ be a smooth solution of the following whole space drift-diffusion equation

$$\partial_t \theta + u \cdot \nabla \theta + \mathcal{L}\theta - \epsilon \Delta \theta = 0, \quad \theta(0, x) = \theta_0(x), \quad x \in \mathbb{R}^d, \quad (3.1)$$

with $\epsilon \geq 0$. Assume that

- (1) for every $t \geq 0$, $\omega(\xi, t)$ is a MOC and satisfies that its inverse function $\omega^{-1}(3\|\theta(\cdot, t)\|_{L_x^\infty}, t) < \infty$;
- (2) for every fixed point ξ , $\omega(\xi, t)$ is piecewise C^1 in the time variable with one-sided derivatives defined at each point, and that for all ξ near infinity, $\omega(\xi, t)$ is continuous in t uniformly in ξ ;
- (3) $\omega(0+, t)$ and $\partial_\xi \omega(0+, t)$ are continuous in t with values in $\mathbb{R} \cup \{\pm\infty\}$, and satisfy that for every $t \geq 0$, either $\omega(0+, t) > 0$ or $\partial_\xi \omega(0+, t) = \infty$ or $\partial_\xi \omega(0+, t) = -\infty$.

Let the initial data $\theta_0(x)$ obey $\omega(\xi, 0)$, then for some $T > 0$, $\theta(x, T)$ obeys the modulus of continuity $\omega(\xi, T)$ provided that for all $t \in]0, T]$ and $\xi \in \{\xi > 0 : \omega(\xi, t) \leq 2\|\theta(\cdot, t)\|_{L_x^\infty}\}$, $\omega(\xi, t)$ satisfies

$$\partial_t \omega(\xi, t) > \Omega_{x,e}(\xi, t) \partial_\xi \omega(\xi, t) + D_{x,e}(\xi, t) + 2\epsilon \partial_\xi \omega(\xi, t), \quad (3.2)$$

where $\Omega_{x,e}(\xi, t)$ and $D_{x,e}(\xi, t)$ are respectively defined from that for every $x \in \mathbb{R}^d$ and every unit vector $e \in \mathbb{S}^{d-1}$ in (3.5),

$$\Omega_{x,e}(\xi, t) := |(u(x + \xi e, t) - u(x, t)) \cdot e|, \quad \text{and} \quad (3.3)$$

$$D_{x,e}(\xi, t) := -(\mathcal{L}\theta(x, t) - \mathcal{L}\theta(x + \xi e, t)), \quad (3.4)$$

under the scenario that

$$\begin{aligned} \theta(x, t) - \theta(x + \xi e, t) &= \omega(\xi, t), \quad \text{and} \\ |\theta(y, t) - \theta(z, t)| &\leq \omega(|y - z|, t), \quad \forall y, z \in \mathbb{R}^d. \end{aligned} \quad (3.5)$$

In (3.2), at the points where $\partial_t \omega(\xi, t)$ (or $\partial_\xi \omega(\xi, t)$) does not exist, the smaller (or larger) value of the one-sided derivative should be taken.

The following lemma is concerned with the estimate of (3.4) under the scenario (3.5).

Lemma 3.3. Assume that the diffusion operator \mathcal{L} is defined by (1.3) with the radial kernel K , then we have the following estimates on $D_{x,e}(\xi, t)$ defined by (3.4) under the scenario (3.5).

- (1) If K satisfies (1.8) with m satisfying (i) and (ii)', then for any $\xi > 0$,

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C_1 \int_0^{\frac{\xi}{2}} (\omega(\xi + 2\eta, t) + \omega(\xi - 2\eta, t) - 2\omega(\xi, t)) \frac{m(\eta^{-1})}{\eta} d\eta \\ &\quad + C_1 \int_{\frac{\xi}{2}}^\infty (\omega(2\eta + \xi, t) - \omega(2\eta - \xi, t) - 2\omega(\xi, t)) \frac{m(\eta^{-1})}{\eta} d\eta, \end{aligned} \quad (3.6)$$

with $C_1 > 0$ a constant depending only on d .

- (2) If K satisfies (1.4)-(1.5) with m satisfying (i)-(ii), then for every $\xi \in]0, \frac{c_0}{2}]$,

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C_1 \int_0^{\frac{\xi}{2}} (\omega(\xi + 2\eta, t) + \omega(\xi - 2\eta, t) - 2\omega(\xi, t)) \frac{m(\eta^{-1})}{\eta} d\eta \\ &\quad + C_1 \int_{\frac{\xi}{2}}^{\frac{c_0}{2}} (\omega(2\eta + \xi, t) - \omega(2\eta - \xi, t) - 2\omega(\xi, t)) \frac{m(\eta^{-1})}{\eta} d\eta. \end{aligned} \quad (3.7)$$

(3) If K satisfies (1.4), (1.7) with m satisfying (i)-(ii), then for every $\xi \in]0, \frac{c_0}{2}]$,

$$D_{x,e}(\xi, t) \leq C'_1 \omega(\xi, t) + R.H.S. \text{ of (3.7)}, \quad (3.8)$$

where $C'_1 > 0$ is a constant depending on d, α, σ and $\tilde{\alpha}$.

Proof of Lemma 3.3. According to (1.3) and (3.5), we see that

$$D_{x,e}(\xi, t) = \int_{\mathbb{R}^d} K(y) (\theta(x+y, t) - \theta(x+\xi e+y, t) - \omega(\xi, t)) dy, \quad (3.9)$$

where the integral will be understood in the sense of principle value if needed. By arguing as the proof of [19, Lemma 2.3], we get

$$\begin{aligned} D_{x,e}(\xi, t) &\leq \int_0^{\frac{\xi}{2}} (\omega(\xi+2\eta, t) + \omega(\xi-2\eta, t) - 2\omega(\xi, t)) \tilde{K}(\eta) d\eta \\ &\quad + \int_{\frac{\xi}{2}}^{\infty} (\omega(2\eta+\xi, t) - \omega(2\eta-\xi, t) - 2\omega(\xi, t)) \tilde{K}(\eta) d\eta, \end{aligned} \quad (3.10)$$

with

$$\tilde{K}(\eta) = \int_{\mathbb{R}^{d-1}} K(\eta, \nu) d\nu. \quad (3.11)$$

(1) If K satisfies (1.8) with m satisfying (i) and (ii)', then by using (2.8), we infer that for every $\eta > 0$,

$$\begin{aligned} \tilde{K}(\eta) &\geq c_1^{-1} \int_{\mathbb{R}^{d-1}} \frac{m((\eta^2 + |\nu|^2)^{-1/2})}{(\eta^2 + |\nu|^2)^{d/2}} d\nu \\ &\geq c_1^{-1} \eta^\alpha m(\eta^{-1}) \int_{\mathbb{R}^{d-1}} \frac{1}{(\eta^2 + |\nu|^2)^{(d+\alpha)/2}} d\nu \\ &\geq c_1^{-1} \frac{m(\eta^{-1})}{\eta} \int_{\mathbb{R}^{d-1}} \frac{1}{(1 + |\nu'|^2)^{(d+\alpha)/2}} d\nu' \geq C_1 \frac{m(\eta^{-1})}{\eta}, \end{aligned} \quad (3.12)$$

where in the last inequality we used

$$c_1^{-1} \int_{\mathbb{R}^{d-1}} \frac{1}{(1 + |\nu'|^2)^{(d+\alpha)/2}} d\nu' \geq c_1^{-1} \int_{|\nu'| \leq 1} \frac{1}{2^{(d+\alpha)/2}} d\nu' \geq c_1^{-1} \frac{1}{2^{(d+1)/2}} |B_1(0)| = C_1.$$

Inserting the above estimate into (3.10) leads to (3.6).

(2) If K satisfies (1.4)-(1.5) with m satisfying (i)-(ii) and $\xi \leq c_0/2$ is concerned, we mainly consider the scope $\eta \in]0, \frac{c_0}{2}]$ and $|\nu| \in]0, \frac{c_0}{2}]$ so that $(\eta^2 + |\nu|^2)^{1/2} \in]0, c_0]$, thus similarly as (3.12), we get that for all $\eta \in]0, \frac{1}{2}]$,

$$\begin{aligned} \tilde{K}(\eta) &\geq c_1^{-1} \int_{\nu \in \mathbb{R}^{d-1}, |\nu| \leq \frac{c_0}{2}} \frac{m((\eta^2 + |\nu|^2)^{-1/2})}{(\eta^2 + |\nu|^2)^{d/2}} d\nu \\ &\geq c_1^{-1} \frac{m(\eta^{-1})}{\eta} \int_{\nu' \in \mathbb{R}^{d-1}, |\nu'| \leq 1} \frac{1}{(1 + |\nu'|^2)^{\frac{d+\alpha}{2}}} d\nu' \geq C_1 \frac{m(\eta^{-1})}{\eta}, \end{aligned}$$

which ensures (3.7).

(3) If K satisfies (1.4), (1.7) with m satisfying (i)-(ii), and $\xi \leq \frac{c_0}{2}$ is concerned, we divide the (η, ν) integral region of the R.H.S. of (3.10) into several parts $\{\eta \in [\frac{c_0}{2}, \infty[\}$, $\{\eta \in]0, \frac{c_0}{2}], |\nu| \in]0, \frac{c_0}{2}]\}$ and $\{\eta \in]0, \frac{c_0}{2}], |\nu| \in [\frac{c_0}{2}, \infty[\}$. The part $\eta \in]0, \frac{c_0}{2}]$ and $|\nu| \in]0, \frac{c_0}{2}]$ can be treated as above and the bound

is the R.H.S. of (3.7). For $\eta \geq \frac{c_0}{2}$, the kernel $K(\eta, \nu)$ may be non-positive, and from (1.7) we infer that

$$\begin{aligned} -\tilde{K}(\eta) &\leq -\int_{(\eta^2+|\nu|^2)^{1/2} \leq c_0} K(\eta, \nu) d\nu - \int_{(\eta^2+|\nu|^2)^{1/2} \geq c_0} K(\eta, \nu) d\nu \\ &\leq c_1 \int_{\mathbb{R}^{d-1}} \frac{1}{(\eta^2 + |\nu|^2)^{\frac{d+\tilde{\alpha}}{2}}} d\nu \leq c_1 \frac{1}{\eta^{1+\tilde{\alpha}}} \int_{\mathbb{R}^{d-1}} \frac{1}{(1 + |\nu'|^2)^{\frac{d+\tilde{\alpha}}{2}}} d\nu' \leq c_1 C_d \frac{1}{\eta^{1+\tilde{\alpha}}}, \end{aligned}$$

and thus the contribution from this part is

$$\begin{aligned} &\int_{\frac{c_0}{2}}^{\infty} (2\omega(\xi, t) + \omega(2\eta - \xi, t) - \omega(2\eta + \xi, t)) (-\tilde{K}(\eta)) d\eta \\ &\leq c_1 C_d 2\omega(\xi, t) \int_{\frac{c_0}{2}}^{\infty} \frac{1}{\eta^{1+\tilde{\alpha}}} d\eta \leq \frac{C'}{2} \omega(\xi, t). \end{aligned}$$

For the part $\eta \in]0, \frac{c_0}{2}]$ and $|\nu| \geq \frac{c_0}{2}$, from (1.7) we get

$$\begin{aligned} -\int_{\nu \in \mathbb{R}^{d-1}, |\nu| \geq \frac{c_0}{2}} K(\eta, \nu) d\nu &\leq -\int_{\nu \in \mathbb{R}^{d-1}, |\nu| \geq \frac{c_0}{2}, (\eta^2+|\nu|^2)^{1/2} \geq c_0} K(\eta, \nu) d\nu \\ &\leq c_1 \int_{\nu \in \mathbb{R}^{d-1}, |\nu| \geq \frac{c_0}{2}} \frac{1}{(\eta^2 + |\nu|^2)^{\frac{d+\tilde{\alpha}}{2}}} d\nu \leq C_{d,\tilde{\alpha}} c_1 c_0^{\tilde{\alpha}}, \end{aligned}$$

and thus the contribution from this part is bounded by

$$\begin{aligned} &C_{d,\tilde{\alpha}} c_1 c_0^{\tilde{\alpha}} \left(\int_0^{\frac{\xi}{2}} (2\omega(\xi, t) - \omega(\xi + 2\eta, t) - \omega(\xi - 2\eta, t)) + \int_{\frac{\xi}{2}}^{\frac{c_0}{2}} (2\omega(\xi, t) + \omega(2\eta - \xi, t) - \omega(2\eta + \xi, t)) \right) \\ &\leq C_{d,\tilde{\alpha}} c_1 c_0^{\tilde{\alpha}} \left(\omega(\xi, t) \frac{\xi}{2} + 2\omega(\xi, t) \frac{c_0 - \xi}{2} \right) \leq \frac{C'_1}{2} \omega(\xi, t). \end{aligned}$$

Hence, gathering the above estimates yields (3.8). \square

Next we consider the estimation of (3.3) under the scenario (3.5).

Lemma 3.4. *Assume that $u = \mathcal{P}(\theta)$ is defined by (1.2), and the diffusion operator \mathcal{L} is given by (1.3) with the radial kernel K , then we have the following estimates on $\Omega_{x,e}(\xi, t)$ under the scenario (3.5).*

(1) *If K satisfies (1.8) with m satisfying (i) and (ii)', then for all $\xi > 0$,*

$$\Omega_{x,e}(\xi, t) \leq -\frac{C_2}{m(\xi^{-1})} D_{x,e}(\xi, t) + C_2 \omega(\xi, t) + C_2 \xi \int_{\xi}^{\infty} \frac{\omega(\eta, t)}{\eta^2} d\eta, \quad (3.13)$$

with $C_2 > 0$ depending only on $d, |a|, |\Psi|$.

(2) *If K satisfies (1.4)-(1.5) with m satisfying (i)-(ii), then we also get (3.13) for all $0 < \xi \leq \frac{c_0}{2}$.*

(3) *If K satisfies (1.4) and (1.7) with m satisfying (i)-(ii), then for all $0 < \xi \leq \frac{c_0}{2}$,*

$$\Omega_{x,e}(\xi, t) \leq -\frac{C_2}{m(\xi^{-1})} D_{x,e}(\xi, t) + (C'_2 + C_2) \omega(\xi, t) + C_2 \xi \int_{\xi}^{\infty} \frac{\omega(\eta, t)}{\eta^2} d\eta, \quad (3.14)$$

with some $C'_2 > 0$ depending on d, α, σ and $|\Psi|$.

(4) *No matter what conditions of K is assumed, there exists a constant $C_3 > 0$ depending only on $d, |a|, |\Psi|$ such that*

$$\Omega_{x,e}(\xi, t) \leq C_3 \omega(\xi, t) + C_3 \int_0^{\xi} \frac{\omega(\eta, t)}{\eta} d\eta + C_3 \xi \int_{\xi}^{\infty} \frac{\omega(\eta, t)}{\eta^2} d\eta. \quad (3.15)$$

Notice that for $\mathcal{L} = |D|^{\alpha}$ and $u = H(\theta)$ with H the 1D Hilbert transform, an estimate similar to (3.13) was obtained in [20, Lemma 2.7].

Proof of Lemma 3.4. For simplicity, we suppress the time variable t in $\omega(\xi, t)$, $\Omega(\xi, t)$ and $D(\xi, t)$. By virtue of (1.2), we see that

$$\begin{aligned} |u(x) - u(x + \xi e)| &= \left| a\omega(\xi) + \text{p.v.} \int_{\mathbb{R}^d} \frac{\Psi(\hat{y})}{|y|^d} \theta(x + y) dy - \text{p.v.} \int_{\mathbb{R}^d} \frac{\Psi(\hat{y})}{|y|^d} \theta(x + \xi e + y) dy \right| \\ &\leq |a|\omega(\xi) + |I(\xi)| + |II(\xi)|, \end{aligned}$$

with $\hat{y} = \frac{y}{|y|} \in \mathbb{S}^{d-1}$, and

$$I(\xi) := \text{p.v.} \int_{|y| \leq 2\xi} \frac{\Psi(\hat{y})}{|y|^d} \theta(x + y) dy - \text{p.v.} \int_{|y| \leq 2\xi} \frac{\Psi(\hat{y})}{|y|^d} \theta(x + \xi e + y) dy, \quad (3.16)$$

$$\text{and } II(\xi) := \int_{|y| \geq 2\xi} \frac{\Psi(\hat{y})}{|y|^d} \theta(x + y) dy - \int_{|y| \geq 2\xi} \frac{\Psi(\hat{y})}{|y|^d} \theta(x + \xi e + y) dy.$$

First we note that the estimation of $II(\xi)$ and the proof of (3.15) are classical, and one can refer to [28, Lemma] or [31, Lemma 3.2] to see that

$$|II(\xi)| \leq C\xi \int_{\xi}^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta, \quad \text{and} \quad |I(\xi)| \leq C \int_0^{\xi} \frac{\omega(\eta)}{\eta} d\eta.$$

Thus for the statements (1)-(3), it suffices to estimate $I(\xi)$ by virtue of $D_{x,e}(\xi)$. Thanks to the zero-average property of $\Psi(\hat{y})$ and the scenario (3.5), we have

$$\begin{aligned} I(\xi) &= \int_{|y| \leq 2\xi} \frac{\Psi(\hat{y})}{|y|^d} (\theta(x + y) - \theta(x)) dy - \int_{|y| \leq 2\xi} \frac{\Psi(\hat{y})}{|y|^d} (\theta(x + \xi e + y) - \theta(x + \xi e)) dy \\ &= \int_{|y| \leq 2\xi} \frac{\Psi(\hat{y})}{|y|^d} (\theta(x + y) - \theta(x + \xi e + y) - \omega(\xi)) dy, \end{aligned}$$

where the integral will be understood in the sense of principle value if needed.

(1) If K satisfies (1.8) with m satisfying (i) and (ii)', recalling that $D_{x,e}(\xi)$ defined by (3.4) has the formula (3.9), and using (2.8)-(2.9), we obtain that for some constant $B > 0$ chosen later,

$$\begin{aligned} I(\xi) + \frac{B}{m(\xi^{-1})} D_{x,e}(\xi) &\leq \int_{|y| \leq 2\xi} \left(\frac{\Psi(\hat{y})}{|y|^d} - c_1^{-1} \frac{B}{m(\xi^{-1})} \frac{m(|y|^{-1})}{|y|^d} \right) (\omega(\xi) + \theta(x + \xi e + y) - \theta(x + y)) dy \\ &\quad - \int_{|y| \geq 2\xi} K(y) (\omega(\xi) + \theta(x + \xi e + y) - \theta(x + y)) dy \\ &\leq \int_{|y| \leq 2\xi} \left(\frac{\Psi(\hat{y})}{|y|^d} - 2^{-\sigma} c_1^{-1} B \frac{\xi^{\alpha-\sigma}}{|y|^{d+\alpha-\sigma}} \right) (\omega(\xi) + \theta(x + \xi e + y) - \theta(x + y)) dy \\ &\leq \int_{|y| \leq 2\xi} (2^{\alpha-\sigma} \Psi(\hat{y}) - 2^{-\sigma} c_1^{-1} B) \frac{\xi^{\alpha-\sigma}}{|y|^{d+\alpha-\sigma}} (\omega(\xi) + \theta(x + \xi e + y) - \theta(x + y)) dy, \end{aligned}$$

where in the third line we used $|y|^{\alpha-\sigma} m(|y|^{-1}) \geq (2\xi)^{\alpha-\sigma} m((2\xi)^{-1}) \geq 2^{-\sigma} \xi^{\alpha-\sigma} m(\xi^{-1})$ for all $0 < |y| \leq 2\xi$. Thus by choosing $B = 2c_1 (\max_{\hat{y} \in \mathbb{S}^{d-1}} |\Psi(\hat{y})|)$, we get

$$|I(\xi)| \leq -\frac{B}{m(\xi^{-1})} D_{x,e}(\xi). \quad (3.17)$$

(2) If K satisfies (1.4)-(1.5) with m satisfying (i)-(ii), and we only consider ξ in the range $0 < \xi \leq \frac{\alpha_0}{2}$, then due to that $K \geq 0$ on all $\mathbb{R}^d \setminus \{0\}$, we similarly obtain (3.17).

(3) If K satisfies (1.4) and (1.7) with m satisfying (i)-(ii), then for the same B as above and for all $0 < \xi \leq c_0/2$,

$$\begin{aligned} I(\xi) + \frac{B}{m(\xi^{-1})} D_{x,e}(\xi) &\leq \frac{B}{m(\xi^{-1})} \int_{|y| \geq 2\xi} (\omega(\xi) + \theta(x + \xi e + y) - \theta(x + y)) (-K(y)) dy \\ &\leq \frac{c_1 B}{m(2/c_0)} \int_{|y| \geq c_0} (\omega(\xi) + \theta(x + \xi e + y) - \theta(x + y)) \frac{1}{|y|^{d+\tilde{\alpha}}} dy. \end{aligned}$$

By arguing as obtaining (3.8), we deduce that

$$I(\xi) + \frac{B}{m(\xi^{-1})} D_{x,e}(\xi) \leq \tilde{C}'_2 \omega(\xi).$$

Therefore, collecting the above estimates leads to the desired results (3.13)-(3.15). \square

4. GLOBAL WELL-POSEDNESS FOR THE SLIGHTLY SUPERCRITICAL CASE

The purpose of this section is to prove Theorem 1.1. We assume $\theta \in C([0, T^*]; H^s) \cap C^\infty([0, T^*] \times \mathbb{R}^d)$, $s > \frac{d}{2} + 1$ is the associated maximal lifespan solution constructed in Theorem 7.1 for the drift-diffusion equation (1.1)-(1.2), and we will show $T^* = \infty$ in the considered cases of Theorem 1.1.

According to the blowup criterion (7.2), it only needs to prove that $\sup_{t \in [0, T^*]} \|\theta(t)\|_{\dot{C}^\beta(\mathbb{R}^d)} < \infty$ for some $\beta > 1 - \alpha + \sigma$. To this end, we will prove that the evolution of the concerned equation (1.1)-(1.2) preserves the following stationary MOC that for $\alpha \in]0, 1]$, $\sigma \in [0, \alpha[$ and $\beta \in]1 - \alpha + \sigma, 1]$,

$$\omega(\xi) = \begin{cases} \kappa m(\delta^{-1}) \delta^{1-\beta} \xi^\beta, & \text{for } 0 < \xi \leq \delta, \\ \kappa m(\delta^{-1}) \delta + \gamma \int_\delta^\xi m(\eta^{-1}) d\eta, & \text{for } \delta < \xi \leq c_{\alpha, \sigma}, \\ \omega(c_{\alpha, \sigma}), & \text{for } \xi > c_{\alpha, \sigma}, \end{cases} \quad (4.1)$$

where $\delta < c_{\alpha, \sigma}$, κ, γ are all positive constants chosen later (κ, γ are independent of δ). In fact, with such a result, and by using (4.5) below, we deduce that

$$\sup_{t \in [0, T^*]} \|\theta(t)\|_{\dot{C}^\beta} = \sup_{t \in [0, T^*]} \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|\theta(x, t) - \theta(y, t)|}{|x - y|^\beta} \leq \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{\omega(|x - y|)}{|x - y|^\beta} \leq \kappa m(\delta^{-1}) \delta^{1-\beta}, \quad (4.2)$$

which is as desired.

First we show that $\omega(\xi)$ is indeed a MOC satisfying the needing properties. Clearly, $\omega(0+) = 0$, $\omega'(0+) = \kappa \beta m(\delta^{-1}) \delta^{1-\beta} \lim_{\xi \rightarrow 0+} \xi^{\beta-1} = \infty$, which satisfies the condition (3) in Proposition 3.2. Observe that for every $0 < \xi < \delta$,

$$\omega'(\xi) = \kappa \beta m(\delta^{-1}) \delta^{1-\beta} \xi^{\beta-1} > 0, \quad \text{and} \quad \omega''(\xi) = -\kappa \beta (1 - \beta) m(\delta^{-1}) \delta^{1-\beta} \xi^{\beta-2} < 0, \quad (4.3)$$

and for every $\delta < \xi \leq c_{\alpha, \sigma}$ (from (1.6)),

$$\omega'(\xi) = \gamma m(\xi^{-1}) > 0, \quad \text{and} \quad \omega''(\xi) = -\gamma \frac{m'(\xi^{-1})}{\xi^2} \leq -\gamma(\alpha - \sigma) \frac{m(\xi^{-1})}{\xi} < 0, \quad (4.4)$$

and $\omega'(\xi) = 0$ for all $\xi > c_{\alpha, \sigma}$, and for $\xi = \delta$,

$$\omega'(\delta-) = \kappa \beta m(\delta^{-1}), \quad \text{and} \quad \omega'(\delta+) = \gamma m(\delta^{-1}),$$

thus if $\gamma < \kappa \beta$, we infer that ω is nondecreasing and concave for all $\xi > 0$ (in fact increasing on $\xi \in]0, c_{\alpha, \sigma}]$). We also find that for every $\xi > 0$,

$$\text{the mapping } \xi \mapsto \frac{\omega(\xi)}{\xi^\beta} \text{ is non-increasing.} \quad (4.5)$$

Indeed, if $\xi \in]0, \delta]$ or $\xi \in]c_{\alpha, \sigma}, \infty[$, (4.5) is a direct consequence of (4.1); while if $\xi \in]\delta, c_{\alpha, \sigma}]$, we have $\left(\frac{\omega(\xi)}{\xi^\beta}\right)' = \frac{\xi\omega'(\xi) - \beta\omega(\xi)}{\xi^{\beta+1}}$, and noticing that by (4.4), $\beta > 1 - \alpha + \sigma$ and $\gamma < \beta\kappa$,

$$(\xi\omega'(\xi) - \beta\omega(\xi))' = \omega'(\xi) + \xi\omega''(\xi) - \beta\omega'(\xi) < (1 - \beta - (\alpha - \sigma))\gamma m(\xi^{-1})m(\xi^{-1}) < 0,$$

and

$$\delta\omega'(\delta+) - \beta\omega(\delta) = \gamma m(\delta^{-1})\delta - \beta\kappa m(\delta^{-1})\delta < 0,$$

we deduce that $\frac{d}{d\xi}\left(\frac{\omega(\xi)}{\xi^\beta}\right) < 0$, which implies (4.5) in the range $\xi \in]\delta, c_{\alpha, \sigma}]$.

Then we prove that under the assumption (1.22), the MOC (4.1) with fixed $\kappa, \gamma > 0$ can be obeyed by the initial data θ_0 for δ small enough. In order to show θ_0 has the MOC (4.1), noting that $|\theta_0(x) - \theta_0(y)| \leq 2\|\theta_0\|_{L^\infty}$, and $|\theta_0(x) - \theta_0(y)| \leq \|\theta_0\|_{\dot{C}^\beta} |x - y|^\beta$, it suffices to prove that

$$\min \left\{ 2\|\theta_0\|_{L^\infty}, \|\theta_0\|_{\dot{C}^\beta} \xi^\beta \right\} < \omega(\xi). \quad (4.6)$$

Denote $a_0 := \left(\frac{2\|\theta_0\|_{L^\infty}}{\|\theta_0\|_{\dot{C}^\beta}}\right)^{1/\beta}$, and if $\xi \geq a_0$, then as long as

$$\omega(a_0) > 2\|\theta_0\|_{L^\infty}, \quad (4.7)$$

we have that (4.6) holds for all $\xi \geq a_0$; while if $\xi \leq a_0$, by virtue of (4.7) and the fact $\frac{\omega(\xi)}{\xi^\beta} \geq \frac{\omega(a_0)}{a_0^\beta}$ which is deduced from (4.5), we also obtain (4.6), as the following deduction shows:

$$\|\theta_0\|_{\dot{C}^\beta} \xi^\beta \leq \|\theta_0\|_{\dot{C}^\beta} \frac{a_0^\beta}{\omega(a_0)} \omega(\xi) \leq \frac{2\|\theta_0\|_{L^\infty}}{\omega(a_0)} \omega(\xi) < \omega(\xi).$$

Now we prove that for every θ_0 , the condition (4.7) can be guaranteed by the assumption (1.22). Indeed, without loss of generality we assume that $a_0 \geq \delta$, then we get

$$\omega(a_0) \geq \gamma \int_\delta^{a_0} m(\eta^{-1}) d\eta \rightarrow \infty, \quad \text{as } \delta \rightarrow 0+, \quad (4.8)$$

hence for δ sufficiently small depending on γ and $\|\theta_0\|_{\dot{C}^\beta \cap L^\infty}$, (4.7) is implied.

Next according to Proposition 3.2, it suffices to show that for all $0 < t < T^*$ and all $\xi > 0$ such that $\omega(\xi) > 2B_0$,

$$\Omega_{x,e}(\xi, t)\omega'(\xi) + D_{x,e}(\xi, t) < 0, \quad (4.9)$$

where B_0 is the bound of $\|\theta(\cdot, t)\|_{L_x^\infty}$ (from Lemma 2.4) given by

$$B_0 := \begin{cases} \|\theta_0\|_{L^\infty}, & \text{if Case (I) is considered,} \\ C(\|\theta_0\|_{L^2 \cap L^\infty}, \alpha, \sigma, d), & \text{if Case (II) is considered,} \end{cases} \quad (4.10)$$

and $\Omega_{x,e}(\xi, t)$, $D_{x,e}(\xi, t)$ are respectively defined by (3.3) and (3.4) under the scenario (3.5) with $\omega(\cdot, t)$ in place of $\omega(\cdot)$. Thanks to (4.8) again, and by letting $b_0 \in]0, \frac{c_{\alpha, \sigma}}{2}]$ be a small constant chosen later (cf. (4.22)), we can also have

$$\omega(b_0) \geq \gamma \int_\delta^{b_0} m(\eta^{-1}) d\eta > 2B_0, \quad (4.11)$$

by choosing δ sufficiently small, thus the scope of ξ we need to treat is just $0 < \xi \leq b_0$. By using Lemma 3.3 and Lemma 3.4, we get

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C_1' \omega(\xi) + C_1 \int_0^{\frac{\xi}{2}} (\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)) \frac{m(\eta^{-1})}{\eta} d\eta \\ &\quad + C_1 \int_{\frac{\xi}{2}}^{\frac{c_{\alpha, \sigma}}{2}} (\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)) \frac{m(\eta^{-1})}{\eta} d\eta, \end{aligned} \quad (4.12)$$

and

$$\Omega_{x,e}(\xi, t) \leq -\frac{C_2}{m(\xi^{-1})} D_{x,e}(\xi, t) + (C'_2 + C_2) \omega(\xi) + C_2 \xi \int_{\xi}^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta, \quad (4.13)$$

where $C_1, C_2 > 0$, and C'_1, C'_2 are just the constants respectively appearing in (3.8) and (3.15) if Case (II) is assumed, and we set $C'_1 = C'_2 = 0$ if Case (I) is assumed.

In order to prove (4.9), we divide into two cases.

Case 1: $0 < \xi \leq \delta$.

In this case, we have $\omega(\xi) = \kappa m(\delta^{-1}) \delta^{1-\beta} \xi^{\beta}$, and $\omega'(\xi) = \kappa \beta m(\delta^{-1}) \delta^{1-\beta} \xi^{\beta-1}$, and from (4.5) we see that

$$\begin{aligned} \int_{\xi}^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta &= \int_{\xi}^{\delta} \frac{\omega(\eta)}{\eta^2} d\eta + \int_{\delta}^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta \\ &= \kappa m(\delta^{-1}) \delta^{1-\beta} \int_{\xi}^{\delta} \eta^{\beta-2} d\eta + \int_{\delta}^{\infty} \frac{\omega(\eta)}{\eta^{\beta}} \frac{1}{\eta^{2-\beta}} d\eta \\ &\leq \kappa m(\delta^{-1}) \delta^{1-\beta} \frac{1}{1-\beta} (\xi^{\beta-1} - \delta^{\beta-1}) + \kappa m(\delta^{-1}) \delta^{1-\beta} \frac{1}{1-\beta} \delta^{\beta-1} \\ &\leq \frac{\kappa}{1-\beta} m(\delta^{-1}) \delta^{1-\beta} \xi^{\beta-1}. \end{aligned}$$

Thus we find

$$\Omega_{x,e}(\xi, t) \omega'(\xi) \leq -\frac{C_2}{m(\xi^{-1})} \omega'(\xi) D_{x,e}(\xi, t) + \frac{2(C_2 + C'_2)}{1-\beta} \left(\kappa m(\delta^{-1}) \delta^{1-\beta} \right)^2 \beta \xi^{2\beta-1}.$$

Observing that for every $\beta > 1 - \alpha + \sigma$ and $\xi \in]0, \delta]$,

$$\frac{C_2}{m(\xi^{-1})} \omega'(\xi) = \frac{C_2 \beta \kappa m(\delta^{-1}) \delta^{1-\beta} \xi^{\beta-1+\alpha-\sigma}}{\xi^{\alpha-\sigma} m(\xi^{-1})} \leq \frac{C_2 \beta \kappa m(\delta^{-1}) \delta^{1-\beta} \delta^{\beta-1+\alpha-\sigma}}{\delta^{\alpha-\sigma} m(\delta^{-1})} = C_2 \beta \kappa,$$

we further get that by letting $\kappa < 1/(2C_2\beta)$,

$$\Omega_{x,e}(\xi, t) \omega'(\xi) \leq -\frac{1}{2} D_{x,e}(\xi, t) + \frac{2(C_2 + C'_2)}{1-\beta} \left(\kappa m(\delta^{-1}) \delta^{1-\beta} \right)^2 \beta \xi^{2\beta-1}. \quad (4.14)$$

For the contribution from the diffusion term, by virtue of the following estimate

$$\begin{aligned} \omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi) &= 4\eta^2 \int_0^1 \int_{-1}^1 \lambda \omega''(\xi + 2\lambda\tau\eta) d\tau d\lambda \\ &\leq 4\eta^2 \int_0^1 \int_{-1}^0 \lambda \omega''(\xi) d\tau d\lambda \leq \omega''(\xi) \eta^2, \end{aligned} \quad (4.15)$$

and (4.3), (2.9), we directly have

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C'_1 \omega(\xi) + C_1 \int_0^{\frac{\xi}{2}} (\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)) \frac{m(\eta^{-1})}{\eta} d\eta \\ &\leq C'_1 \omega(\xi) + C_1 \omega''(\xi) \int_0^{\frac{\xi}{2}} \eta m(\eta^{-1}) d\eta \\ &\leq C'_1 \omega(\xi) - C_1 \beta (1-\beta) \kappa m(\delta^{-1}) \delta^{1-\beta} \xi^{\beta-2} \int_0^{\frac{\xi}{2}} (\eta^{\alpha-\sigma} m(\eta^{-1})) \eta^{1-\alpha+\sigma} d\eta \\ &\leq C'_1 \kappa m(\delta^{-1}) \delta^{1-\beta} \xi^{\beta} - \frac{C_1}{8} \beta (1-\beta) \kappa (m(\delta^{-1}))^2 \delta^{1-\beta+\alpha-\sigma} \xi^{\beta-\alpha+\sigma} \\ &\leq -\frac{C_1}{16} \beta (1-\beta) \kappa (m(\delta^{-1}))^2 \delta^{1-\beta+\alpha-\sigma} \xi^{\beta-\alpha+\sigma}, \end{aligned} \quad (4.16)$$

where the last inequality is guaranteed by setting $m(\delta^{-1}) \geq \frac{16C'_1}{C_1\beta(1-\beta)}$, or setting

$$\delta \leq \left(\frac{C_1\beta(1-\beta)}{16C'_1} \right)^{\frac{1}{\alpha-\sigma}}. \quad (4.17)$$

Hence we infer that

$$\begin{aligned} & \Omega_{x,e}(\xi, t)\omega'(\xi) + D_{x,e}(\xi, t) \leq \\ & \leq \frac{2(C_2 + C'_2)}{1-\beta} \beta \left(\kappa m(\delta^{-1})\delta^{1-\beta} \right)^2 \xi^{2\beta-1} + \frac{1}{2} D_{x,e}(\xi, t) \\ & \leq \beta \kappa \left(m(\delta^{-1}) \right)^2 \delta^{1-\beta+\alpha-\sigma} \xi^{\beta-\alpha+\sigma} \left(\frac{2(C_2 + C'_2)}{1-\beta} \kappa \left(\frac{\xi}{\delta} \right)^{\beta-1+\alpha-\sigma} - \frac{C_1(1-\beta)}{32} \right) \\ & \leq \beta \kappa \left(m(\delta^{-1}) \right)^2 \delta^{1-\beta+\alpha-\sigma} \xi^{\beta-\alpha+\sigma} \left(\frac{2(C_2 + C'_2)}{1-\beta} \kappa - \frac{C_1(1-\beta)}{32} \right) < 0, \end{aligned} \quad (4.18)$$

where the last inequality is from choosing κ so that $\kappa < \frac{C_1}{64(C_2+C'_2)}(1-\beta)^2$.

Case 2: $\delta < \xi \leq b_0$.

Taking advantage of (4.5), we have

$$\int_{\xi}^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta = \int_{\xi}^{\infty} \frac{\omega(\eta)}{\eta^{\beta}} \frac{1}{\eta^{2-\beta}} d\eta \leq \frac{\omega(\xi)}{\xi^{\beta}} \int_{\xi}^{\infty} \frac{1}{\eta^{2-\beta}} d\eta \leq \frac{1}{1-\beta} \frac{\omega(\xi)}{\xi}.$$

Thus from (4.13) and $\omega'(\xi) = \gamma m(\xi^{-1})$ in this case, we obtain that by choosing $\gamma < 1/(2C_2)$,

$$\Omega_{x,e}(\xi, t)\omega'(\xi) = -\gamma C_2 D_{x,e}(\xi, t) + \frac{2(C_2 + C'_2)}{1-\beta} \gamma m(\xi^{-1})\omega(\xi) \leq -\frac{1}{2} D_{x,e}(\xi, t) + \frac{2(C_2 + C'_2)}{1-\beta} \gamma m(\xi^{-1})\omega(\xi).$$

For $D_{x,e}(\xi, t)$, noticing that $\omega(2\eta + \xi) - \omega(2\eta - \xi) \leq \omega(2\xi) < 2\omega(\xi)$ and $\xi^{-1} \geq b_0^{-1}$, we get

$$\begin{aligned} D_{x,e}(\xi, t) & \leq C'_1 \omega(\xi) + C_1 (\omega(2\xi) - 2\omega(\xi)) \int_{\frac{\xi}{2}}^{b_0} \frac{m(\eta^{-1})}{\eta} d\eta \\ & \leq C'_1 \omega(\xi) + C_1 (\omega(2\xi) - 2\omega(\xi)) 2^{-\sigma} \xi^{\alpha} m(\xi^{-1}) \frac{1}{\alpha} \left(\left(\frac{2}{\xi} \right)^{\alpha} - b_0^{-\alpha} \right) \\ & \leq C'_1 \omega(\xi) + C_1 \frac{2^{-\sigma}(2^{\alpha} - 1)}{\alpha} (\omega(2\xi) - 2\omega(\xi)) m(\xi^{-1}) \\ & \leq C'_1 \omega(\xi) + \frac{C_1 \tilde{c}}{2} (\omega(2\xi) - 2\omega(\xi)) m(\xi^{-1}), \end{aligned} \quad (4.19)$$

with \tilde{c} defined by

$$\tilde{c} := \inf_{x \in [0,1]} \left\{ \frac{2^x - 1}{x} \right\} > 0. \quad (4.20)$$

Next we claim that for γ small enough, we have

$$\omega(2\xi) \leq \max \{ 2^{1-\alpha+\sigma}, 3/2 \} \omega(\xi), \quad \forall \xi \in [\delta, b_0]. \quad (4.21)$$

Indeed, for $\xi = \delta$, we see that $\omega(\delta) = \kappa m(\delta^{-1})\delta$ and

$$\begin{aligned} \omega(2\delta) & = \omega(\delta) + \gamma \int_{\delta}^{2\delta} m(\eta^{-1}) d\eta \leq \omega(\delta) + \gamma \delta^{\alpha-\sigma} m(\delta^{-1}) \int_{\delta}^{2\delta} \frac{1}{\eta^{\alpha-\sigma}} d\eta \\ & \leq \kappa m(\delta^{-1})\delta + \gamma \delta^{\alpha-\sigma} m(\delta^{-1}) \frac{1}{1-\alpha+\sigma} ((2\delta)^{1-\alpha+\sigma} - \delta^{1-\alpha+\sigma}) \\ & \leq \kappa m(\delta^{-1})\delta + \frac{\gamma}{1-\alpha+\sigma} (2^{1-\alpha+\sigma} - 1) m(\delta^{-1})\delta, \end{aligned}$$

which further yields that for all $\gamma < \frac{\kappa}{2}$,

$$\begin{aligned} \omega(2\delta) &\leq \begin{cases} \kappa m(\delta^{-1})\delta + 2\gamma(2^{1-\alpha+\sigma} - 1)m(\delta^{-1})\delta, & \text{if } \alpha - \sigma \leq 1/2, \\ \kappa m(\delta^{-1})\delta + \gamma \left(\sup_{x \in]0, 1/2]} \frac{2^x - 1}{x} \right) m(\delta^{-1})\delta, & \text{if } \alpha - \sigma > 1/2, \end{cases} \\ &\leq \max \{2^{1-\alpha+\sigma}, 3/2\} \omega(\delta), \end{aligned}$$

where we have used $\sup_{x \in]0, 1/2]} \frac{2^x - 1}{x} \leq \max \left\{ \lim_{x \rightarrow 0^+} \frac{2^x - 1}{x}, \frac{2^{1/2} - 1}{1/2} \right\} \leq 1$. Whereas for $\xi \in]\delta, b_0]$ with $b_0 \in]0, \frac{c_{\alpha, \sigma}}{2}[$ some constant, considering an auxiliary function

$$h(\xi) := \omega(2\xi) - \max \{2^{1-\alpha+\sigma}, 3/2\} \omega(\xi),$$

and noting that

$$h'(\xi) \leq 2\omega'(2\xi) - 2^{1-\alpha+\sigma}\omega'(\xi) = 2m((2\xi)^{-1}) - 2^{1-\alpha+\sigma}m(\xi^{-1}) \leq 0,$$

we deduce $h(\xi) \leq h(\delta) \leq 0$ for all $\xi \geq \delta$, which implies (4.21). Hence, plugging (4.21) into (4.19) leads to

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C'_1 \omega(\xi) - \frac{C_1 \tilde{c}}{2} (2 - \max \{2^{1-\alpha+\sigma}, 3/2\}) m(\xi^{-1}) \omega(\xi) \\ &\leq \omega(\xi) \left(C'_1 - \frac{C_1 \tilde{c}}{2} (1 - 2^{-\alpha+\sigma}) m(\xi^{-1}) \right) \\ &\leq \omega(\xi) \left(C'_1 - \frac{C_1 \tilde{c}^2}{4} (\alpha - \sigma) m(\xi^{-1}) \right) \leq -\frac{C_1 \tilde{c}^2 (\alpha - \sigma)}{8} m(\xi^{-1}) \omega(\xi), \end{aligned}$$

where \tilde{c} is given by (4.20), and in the last inequality we used $m(b_0^{-1}) \geq \frac{8C'_1}{C_1 \tilde{c}^2 (\alpha - \sigma)}$, which can be implied by setting

$$b_0 \leq \left(\frac{C_1 \tilde{c}^2 (\alpha - \sigma)}{8C'_1} \right)^{\frac{1}{\alpha - \sigma}}. \quad (4.22)$$

Collecting the above estimates yields that for all $\xi \in]\delta, b_0]$,

$$\Omega_{x,e}(\xi, t) \omega'(\xi) + D_{x,e}(\xi, t) \leq \left(\frac{2(C_2 + C'_2)}{1 - \beta} \gamma - \frac{C_1 \tilde{c}^2 (\alpha - \sigma)}{8} \right) m(\xi^{-1}) \omega(\xi) < 0, \quad (4.23)$$

where the last inequality is guaranteed as long as γ is satisfying $\gamma < \frac{C_1 \tilde{c}^2 (1 - \beta) (\alpha - \sigma)}{16(C_2 + C'_2)}$.

Therefore, thanks to (4.18) and (4.23), we prove (4.2) for every $\beta \in]1 - \alpha + \sigma, 1[$ with each $\alpha \in]0, 1]$ and $\sigma \in [0, \alpha]$, where κ, γ are some fixed positive constants satisfying

$$\kappa < \min \left\{ \frac{1}{2C_2\beta}, \frac{C_1(1 - \beta)^2}{64(C_2 + C'_2)} \right\}, \quad \gamma < \min \left\{ \beta\kappa, \frac{\kappa}{2}, \frac{1}{2C_2}, \frac{C_1 \tilde{c}^2 (1 - \beta) (\alpha - \sigma)}{16(C_2 + C'_2)} \right\}, \quad (4.24)$$

and $\delta > 0$ is a small constant satisfying (4.8), (4.11) and (4.17) with $b_0 = \min \left\{ \frac{c_{\alpha, \sigma}}{2}, \left(\frac{C_1 \tilde{c}^2 (\alpha - \sigma)}{16C'_1} \right)^{1/(\alpha - \sigma)} \right\}$.

Thus we finish the proof of Theorem 1.1.

5. EVENTUAL UNIFORM-IN- ϵ HÖLDER ESTIMATE OF THE ϵ -REGULARIZED SOLUTION

The purpose of this section is to show Proposition 1.3, and the main method is still the nonlocal maximum principle.

We consider the following family of moduli of continuity that for $\xi_0 > \delta$,

$$\omega(\xi, \xi_0) = \begin{cases} (1 - \beta)\kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi_0} m(\eta^{-1})d\eta - \gamma m(\xi_0^{-1})(\xi_0 - \delta) + \beta\kappa m(\delta^{-1})\xi, & \text{for } 0 < \xi \leq \delta, \\ \kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi_0} m(\eta^{-1})d\eta - \gamma m(\xi_0^{-1})\xi_0 + \gamma m(\xi_0^{-1})\xi, & \text{for } \delta < \xi \leq \xi_0, \\ \kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi} m(\eta^{-1})d\eta, & \text{for } \xi > \xi_0, \end{cases} \quad (5.1)$$

and for $\xi_0 \leq \delta$,

$$\omega(\xi, \xi_0) = \begin{cases} (1 - \beta)\kappa m(\delta^{-1})\delta^{1-\beta}\xi_0^{\beta} + \beta\kappa m(\delta^{-1})\delta^{1-\beta}\xi_0^{\beta-1}\xi, & \text{for } 0 < \xi \leq \xi_0, \\ \kappa m(\delta^{-1})\delta^{1-\beta}\xi^{\beta}, & \text{for } \xi_0 < \xi \leq \delta, \\ \kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi} m(\eta^{-1})d\eta, & \text{for } \xi > \delta, \end{cases} \quad (5.2)$$

where $\beta \in]1 - \alpha + \sigma, 1[$, and κ, γ, δ are positive constants chosen later. Note that for $\xi_0 = 0+$, $\omega(\xi, 0+)$ just reduces to the MOC (4.1) with $c_0 = c_{\alpha, \sigma} = \infty$. Motivated by [25], the basic idea of constructing $\omega(\xi, \xi_0)$ is through taking a tangent line at $\xi = \xi_0$ to $\omega(\xi)$ given by (4.1) and replacing $\omega(\xi)$ with this tangent line at the range $0 < \xi \leq \xi_0$. But since the one-sided derivatives of $\omega(\xi)$ at the point $\xi = \delta$ do not coincide, thus in order to control $\partial_{\xi_0}\omega(\xi, \xi_0)$ at the point $\xi_0 = \delta$, we make a modification in the case $\xi_0 > \delta$, that is, the tangent line mentioned above at the range $\delta \leq \xi \leq \xi_0$ is still adopted, but at the range $0 < \xi \leq \delta$ it is replaced by a straight line crossing $\omega(\delta+, \xi_0)$ with the larger slope $\omega'(\delta-) = \beta\kappa m(\delta^{-1})$.

Clearly, for all $\xi_0 > 0$, $\omega(0+, \xi_0) > 0$, which guarantees the condition (3) in Proposition 3.2. Similarly as $\omega(\xi)$ defined by (4.1), $\omega(\xi, \xi_0)$ is also a increasing and concave function for all $\xi > 0$ and $\xi_0 > 0$. For $\xi_0 = A_0 > \delta$, by virtue of (2.9), we get

$$\begin{aligned} \omega(0+, A_0) &= (1 - \beta)\kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{A_0} m(\eta^{-1})d\eta - \gamma m(A_0^{-1})(A_0 - \delta) \\ &\geq (1 - \beta)\kappa m(\delta^{-1})\delta + \gamma m(A_0^{-1}) A_0^{\alpha-\sigma} \int_{\delta}^{A_0} \eta^{-(\alpha-\sigma)}d\eta - \gamma m(A_0^{-1})A_0 \\ &\geq (1 - \beta)\kappa m(\delta^{-1})\delta + \frac{\gamma}{1 - \alpha + \sigma} m(A_0^{-1}) A_0^{\alpha-\sigma} (A_0^{1-\alpha+\sigma} - \delta^{1-\alpha-\sigma}) - \gamma m(A_0^{-1})A_0 \\ &\geq ((1 - \beta)\kappa - \gamma)m(\delta^{-1})\delta + \frac{(\alpha - \sigma)\gamma}{1 - \alpha + \sigma} m(A_0^{-1})A_0^{\alpha-\sigma} (A_0^{1-\alpha+\sigma} - \delta^{1-\alpha+\sigma}). \end{aligned} \quad (5.3)$$

Since we assume $\gamma < (1 - \beta)\kappa$, thus we have that the initial data θ_0 obeys the MOC $\omega(\xi, A_0)$ provided that

$$\frac{(\alpha - \sigma)\gamma}{1 - \alpha + \sigma} m(A_0^{-1})A_0^{\alpha-\sigma} (A_0^{1-\alpha+\sigma} - \delta^{1-\alpha+\sigma}) \geq 2\|\theta_0\|_{L^\infty}. \quad (5.4)$$

According to Proposition 2.5, we next will prove the following lemma.

Lemma 5.1. *Suppose that Case (III) is considered, and the initial data θ_0 obeys the MOC $\omega(\xi, A_0)$ given by (5.1). For $\rho > 0$, let $\xi_0 = \xi_0(t)$ be a function satisfying*

$$\frac{d}{dt}\xi_0 = -\rho m(\xi_0^{-1})\xi_0, \quad \xi_0(0) = A_0. \quad (5.5)$$

Then for some positive constants $\delta, \kappa, \gamma, \rho$ small enough, the solution $\theta(x, t)$ of the regularized drift-diffusion equation (1.24) obeys the MOC $\omega(\xi, \xi_0(t))$ for all t such that $\xi_0(t) \geq 0$.

Now with Lemma 5.1 at our disposal (whose proof is postponed later), we can conclude Proposition 1.3 as follows. Thanks to (5.5), and by integrating on the time variable over $[0, t]$, we get

$$\begin{aligned} \rho t &= \int_{\xi_0(t)}^{A_0} \frac{1}{m(\xi_0^{-1})\xi_0} d\xi_0 \leq \frac{1}{A_0^{\alpha-\sigma}m(A_0^{-1})} \int_{\xi_0(t)}^{A_0} \frac{1}{\xi_0^{1-\alpha+\sigma}} d\xi_0 \\ &= \frac{1}{A_0^{\alpha-\sigma}m(A_0^{-1})} \frac{1}{\alpha-\sigma} (A_0^{\alpha-\sigma} - \xi_0(t)^{\alpha-\sigma}), \end{aligned}$$

which yields that

$$\xi_0(t) \leq A_0 (1 - m(A_0^{-1})(\alpha - \sigma)\rho t)^{\frac{1}{\alpha-\sigma}}. \quad (5.6)$$

Thus there exists a time t_1 satisfying

$$t_1 \leq \frac{1}{(\alpha - \sigma)\rho m(A_0^{-1})}, \quad (5.7)$$

so that $\xi_0(t_1) \equiv 0$ and $\theta(x, t_1)$ obeys the MOC $\omega(\xi, 0+) = \omega(\xi)$ with $\omega(\xi)$ defined by (4.1). In a similar manner as proving (4.9), we can also show that

$$\Omega_{x,e}(\xi, t)\omega'(\xi) + D_{x,e}(\xi, t) + 2\epsilon\omega''(\xi) < 0, \quad (5.8)$$

for all $t_1 < t < \infty$ and all $\xi > 0$, where $\Omega_{x,e}(\xi, t)$ is given by (4.13) and $D_{x,e}(\xi, t)$ is given by (4.12) with $\frac{\epsilon_0}{2}$ in the second integral replaced by ∞ . Note that compared with the proof of (4.9), we have $C'_1 = C'_2 = 0$ and $b_0 = \infty$, and the conditions on κ, γ are

$$\kappa < \min \left\{ \frac{1}{2C_2\beta}, \frac{C_1(1-\beta)^2}{32C_2} \right\}, \quad \gamma < \min \left\{ \beta\kappa, \frac{\kappa}{2}, \frac{1}{2C_2}, \frac{C_1\tilde{c}^2(1-\beta)(\alpha-\sigma)}{8C_2} \right\}. \quad (5.9)$$

Hence, (5.8) and Proposition 3.2 guarantee that the MOC $\omega(\xi)$ given by (4.1) is preserved by the solution $\theta(x, t)$ for all $t \geq t_1$, and in a similar way as deriving (4.2), we prove that

$$\sup_{t \in [t_1, \infty[} \|\theta(t)\|_{\dot{C}^\beta} \leq \kappa m(\delta^{-1})\delta^{1-\beta}, \quad (5.10)$$

with some fixed $\delta > 0$ satisfying (5.4), and thus we conclude the proof of (1.25).

In particular, if $\alpha \in]0, 1[$ and $\sigma = 0$ in the condition (1.9), then (5.4), (5.7) and (5.10) reduce to

$$\begin{cases} \frac{\alpha\gamma}{1-\alpha} (A_0^{1-\alpha} - \delta^{1-\alpha}) \geq 2\|\theta_0\|_{L^\infty}, \\ t_1 \leq A_0^\alpha/(\alpha\rho), \\ \sup_{t \in [t_1, \infty[} \|\theta(t)\|_{\dot{C}^\beta} \leq \kappa\delta^{1-\alpha-\beta}, \end{cases}$$

where κ, γ, ρ are fixed positive constants satisfying (5.56) below, that is, we can choose

$$\rho = \frac{1-\beta}{C\alpha}, \quad \kappa = \frac{1}{C}(1-\beta)^2, \quad \gamma = \frac{1}{C} \min \{ (1-\beta)^3\alpha, \beta(1-\beta)^2 \}, \quad (5.11)$$

with some $C > 0$ depending on C_1, C_2 . By choosing

$$A_0 = \left(\frac{4(1-\alpha)}{\alpha\gamma} \|\theta_0\|_{L^\infty} \right)^{\frac{1}{1-\alpha}}, \quad \delta = \left(\frac{(1-\alpha)}{\alpha\gamma} \|\theta_0\|_{L^\infty} \right)^{\frac{1}{1-\alpha}},$$

we see that

$$t_1 \leq \frac{C}{1-\beta} \left(\frac{4(1-\alpha)}{\alpha\gamma} \right)^{\frac{\alpha}{1-\alpha}} \|\theta_0\|_{L^\infty}^{\frac{\alpha}{1-\alpha}}, \quad (5.12)$$

and for every $\beta \in]1-\alpha, 1[$, we have

$$\sup_{t \in [t_1, \infty[} \|\theta(t)\|_{\dot{C}^\beta(\mathbb{R}^d)} \leq \frac{(1-\beta)^2}{C} \left(\frac{1-\alpha}{\alpha\gamma} \right)^{-\frac{\beta-1+\alpha}{1-\alpha}} \|\theta_0\|_{L^\infty}^{-\frac{\beta-1+\alpha}{1-\alpha}}, \quad (5.13)$$

where $C > 0$ is some constant depending only on d , and thus finish the proof of Proposition 1.3.

Then the remaining work is to show Lemma 5.1.

Proof of Lemma 5.1. Taking advantage of Proposition 3.2, it suffices to prove that for all $t > 0$ and $\xi > 0$,

$$-\partial_{\xi_0}\omega(\xi, \xi_0)\dot{\xi}_0(t) + \Omega_{x,e}(\xi, t)\partial_{\xi}\omega(\xi, \xi_0) + D_{x,e}(\xi, t) + \epsilon\partial_{\xi\xi}\omega(\xi, \xi_0) < 0, \quad (5.14)$$

where $\omega(\xi, \xi_0)$ is given by (5.1)-(5.2) and

$$\begin{aligned} D_{x,e}(\xi, t) \leq & C_1 \int_0^{\frac{\xi}{2}} (\omega(\xi + 2\eta, \xi_0) + \omega(\xi - 2\eta, \xi_0) - 2\omega(\xi, \xi_0)) \frac{m(\eta^{-1})}{\eta} d\eta \\ & + C_1 \int_{\frac{\xi}{2}}^{\infty} (\omega(2\eta + \xi, \xi_0) - \omega(2\eta - \xi, \xi_0) - 2\omega(\xi, \xi_0)) \frac{m(\eta^{-1})}{\eta} d\eta, \end{aligned} \quad (5.15)$$

and

$$\Omega_{x,e}(\xi, t) \leq -\frac{C_2}{m(\xi^{-1})} D_{x,e}(\xi, t) + C_2\omega(\xi, \xi_0) + C_2\xi \int_{\xi}^{\infty} \frac{\omega(\eta, \xi_0)}{\eta^2} d\eta. \quad (5.16)$$

In (5.14), if $\partial_{\xi_0}\omega(\xi, \xi_0)$ or $\partial_{\xi}\omega(\xi, \xi_0)$ does not exist, the larger value of the one-sided derivative should be taken.

We divide into several cases to prove (5.14), according to the values of ξ_0 and ξ .

Case 1: $\xi_0 > \delta$, $0 < \xi \leq \delta$.

From $\omega(\xi, \xi_0) = (1 - \beta)\kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi_0} m(\eta^{-1})d\eta - \gamma m(\xi_0^{-1})(\xi_0 - \delta) + \beta\kappa m(\delta^{-1})\xi$ in this case, we have

$$\partial_{\xi_0}\omega(\xi, \xi_0) = \gamma\xi_0^{-2}m'(\xi_0^{-1})(\xi_0 - \delta) \leq \gamma\alpha m(\xi_0^{-1}), \quad \text{and} \quad \partial_{\xi}\omega(\xi, \xi_0) = \beta\kappa m(\delta^{-1}), \quad (5.17)$$

and

$$\begin{aligned} \omega(\xi, \xi_0) & \geq \omega(0+, \xi_0) = (1 - \beta)\kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi_0} m(\eta^{-1})d\eta - \gamma m(\xi_0^{-1})(\xi_0 - \delta) \\ & \geq (1 - \beta)\kappa m(\delta^{-1})\delta + \gamma\xi_0^{\alpha-\sigma}m(\xi_0^{-1}) \int_{\delta}^{\xi_0} \frac{1}{\eta^{\alpha-\sigma}}d\eta - \gamma m(\xi_0^{-1})(\xi_0 - \delta) \\ & = (1 - \beta)\kappa m(\delta^{-1})\delta + \frac{\gamma}{1 - \alpha + \sigma}m(\xi_0^{-1})\xi_0^{\alpha-\sigma}(\xi_0^{1-\alpha+\sigma} - \delta^{1-\alpha+\sigma}) - \gamma m(\xi_0^{-1})(\xi_0 - \delta) \\ & =: M_{\xi_0, \delta}, \end{aligned} \quad (5.18)$$

and

$$\omega(\xi, \xi_0) - \omega(0+, \xi_0) \leq \omega(\delta, \xi_0) - \omega(0+, \xi_0) = \beta\kappa m(\delta^{-1})\delta. \quad (5.19)$$

Thus by using (5.5) and (5.17), we get

$$-\partial_{\xi_0}\omega(\xi, \xi_0)\dot{\xi}_0(t) \leq \rho\alpha\gamma (m(\xi_0^{-1}))^2 \xi_0. \quad (5.20)$$

According to (5.1), we obtain

$$\begin{aligned} & \int_{\xi}^{\infty} \frac{\omega(\eta, \xi_0)}{\eta^2} d\eta = \frac{\omega(\xi, \xi_0)}{\xi} + \int_{\xi}^{\infty} \frac{\partial_{\eta}\omega(\eta, \xi_0)}{\eta} d\eta \\ & = \frac{\omega(\xi, \xi_0)}{\xi} + \int_{\xi}^{\delta} \frac{\kappa\beta m(\delta^{-1})}{\eta} d\eta + \int_{\delta}^{\xi_0} \frac{\gamma m(\xi_0^{-1})}{\eta} d\eta + \int_{\xi_0}^{\infty} \frac{\gamma m(\eta^{-1})}{\eta} d\eta \\ & \leq \frac{\omega(\xi, \xi_0)}{\xi} + \kappa\beta m(\delta^{-1}) \left(\log \frac{\delta}{\xi} \right) + \gamma m(\xi_0^{-1}) \left(\log \frac{\xi_0}{\delta} \right) + \gamma\xi_0^{\alpha-\sigma}m(\xi_0^{-1}) \int_{\xi_0}^{\infty} \frac{1}{\eta^{1-\alpha+\sigma}} d\eta \\ & \leq \frac{\omega(\xi, \xi_0)}{\xi} + \kappa\beta m(\delta^{-1}) \left(\log \frac{\delta}{\xi} \right) + \gamma m(\xi_0^{-1}) \left(\log \frac{\xi_0}{\delta} \right) + \frac{\gamma}{\alpha - \sigma}m(\xi_0^{-1}). \end{aligned} \quad (5.21)$$

Thus by using (5.16), (5.19) and (5.21), we find that for $\kappa \leq \frac{1}{4C_2\beta}$,

$$\begin{aligned}
& \Omega_{x,e}(\xi, t) \partial_\xi \omega(\xi, \xi_0) \\
& \leq -C_2\beta\kappa D_{x,e}(\xi, t) + C_2\beta\kappa m(\delta^{-1}) \left(2\omega(\xi, \xi_0) + \kappa\beta m(\delta^{-1})\xi \left(\log \frac{\delta}{\xi} \right) + \gamma m(\xi_0^{-1})\xi \left(\log \frac{\xi_0}{\delta} + \frac{1}{\alpha - \sigma} \right) \right) \\
& \leq -C_2\beta\kappa D_{x,e}(\xi, t) + C_2\beta\kappa m(\delta^{-1}) \left(2\omega(0+, \xi_0) + (C_0 + 2)\kappa\beta m(\delta^{-1})\delta + \gamma m(\xi_0^{-1})\xi_0 \left(C_0 + \frac{1}{\alpha - \sigma} \right) \right) \\
& \leq -\frac{1}{4}D_{x,e}(\xi, t) + C_2(C_0 + 2)\beta^2\kappa^2(m(\delta^{-1}))^2\delta + \frac{C_2\beta(C_0 + 1)\kappa\gamma}{\alpha - \sigma}m(\delta^{-1})m(\xi_0^{-1})\xi_0 + \\
& \quad + 2C_2\beta\kappa m(\delta^{-1})\omega(0+, \xi_0),
\end{aligned} \tag{5.22}$$

where in the third line we also used $\frac{\xi}{\delta} \left(\log \frac{\delta}{\xi} \right) \leq C_0$ and $\frac{\xi_0}{\delta} \log \frac{\xi_0}{\delta} \leq C_0$. For the contribution from the diffusion term, since the function $\omega(\eta, \xi_0) - \omega(0+, \xi_0)$ is still concave, we infer that

$$\begin{aligned}
D_{x,e}(\xi, t) & \leq -2C_1\omega(0+, \xi_0) \int_{\frac{\xi}{2}}^{\infty} \frac{m(\eta^{-1})}{\eta} d\eta \\
& \leq -2C_1\omega(0+, \xi_0) \left(\frac{\xi}{2} \right)^\alpha m\left(\frac{2}{\xi} \right) \int_{\frac{\xi}{2}}^{\infty} \frac{1}{\eta^{1+\alpha}} d\eta \\
& \leq -\frac{2C_1}{\alpha} \omega(0+, \xi_0) m(\xi^{-1}),
\end{aligned} \tag{5.23}$$

and also by (5.18),

$$D_{x,e}(\xi, t) \leq -\frac{2C_1}{\alpha} M_{\xi_0, \delta} m(\xi^{-1}). \tag{5.24}$$

If $\xi_0 \geq N\delta$ with $N \in \mathbb{N}$ a suitable constant, we see that

$$\frac{1}{1 - \alpha + \sigma} (\xi_0^{1-\alpha+\sigma} - \delta^{1-\alpha+\sigma}) \geq \frac{1 - (1/N)^{1-\alpha+\sigma}}{1 - \alpha + \sigma} \xi_0^{1-\alpha+\sigma} \geq \frac{1}{1 - \frac{\alpha - \sigma}{2}} \xi_0^{1-\alpha+\sigma},$$

provided that $1 - (1/N)^{1-\alpha+\sigma} \geq \frac{2(1-\alpha+\sigma)}{2-\alpha+\sigma}$, that is, $N \geq \left(\frac{2-(\alpha-\sigma)}{\alpha-\sigma} \right)^{\frac{1}{1-(\alpha-\sigma)}}$, thus we may choose

$$N := \left\lceil \left(\frac{2 - (\alpha - \sigma)}{\alpha - \sigma} \right)^{\frac{1}{1-(\alpha-\sigma)}} \right\rceil + 1. \tag{5.25}$$

Thus for the case $\xi_0 \geq N\delta$, we get

$$\begin{aligned}
M_{\xi_0, \delta} & \geq (1 - \beta)\kappa m(\delta^{-1})\delta + \left(\frac{1}{1 - \frac{\alpha - \sigma}{2}} - 1 \right) \gamma m(\xi_0^{-1})\xi_0 \\
& \geq (1 - \beta)\kappa m(\delta^{-1})\delta + \gamma(\alpha - \sigma)m(\xi_0)\xi_0.
\end{aligned} \tag{5.26}$$

Inserting the above estimate into (5.24) leads to

$$D_{x,e}(\xi, t) \leq -\frac{2C_1(1 - \beta)\kappa}{\alpha} m(\delta^{-1})\delta m(\xi^{-1}) - \frac{2C_1(\alpha - \sigma)\gamma}{\alpha} m(\xi_0^{-1})\xi_0 m(\xi^{-1}). \tag{5.27}$$

Hence for $\xi_0 \geq N\delta$ with N satisfying (5.25), by (5.23) and setting $\kappa \leq \frac{C_1}{4C_2\beta\alpha}$ so that

$$2C_2\beta\kappa m(\delta^{-1})\omega(0+, \xi_0) \leq \frac{C_1}{2\alpha} m(\xi^{-1})\omega(0+, \xi_0) \leq -\frac{1}{4}D_{x,e}(\xi, t), \tag{5.28}$$

and by collecting (5.20), (5.22) and (5.27), we deduce that

$$\begin{aligned} \text{L.H.S. of (5.14)} &\leq \kappa m(\delta^{-1})\delta m(\xi^{-1}) \left(C_2 (C_0 + 2) \beta^2 \kappa - \frac{C_1(1-\beta)}{\alpha} \right) + \\ &\quad + \gamma m(\xi_0^{-1})\xi_0 m(\xi^{-1}) \left(\rho\alpha + \frac{C_2\beta(C_0+1)}{\alpha-\sigma}\kappa - \frac{C_1(\alpha-\sigma)}{\alpha} \right) < 0, \end{aligned}$$

where the last inequality is guaranteed as long as ρ, κ satisfy

$$\rho < \frac{C_1(\alpha-\sigma)}{2\alpha^2}, \quad \kappa < \min \left\{ \frac{1}{4C_2\beta}, \frac{C_1}{4C_2\beta\alpha}, \frac{C_1(\alpha-\sigma)^2}{2C_2(C_0+1)\beta\alpha}, \frac{C_1(1-\beta)}{C_2(C_0+2)\beta^2\alpha} \right\}. \quad (5.29)$$

If $\xi_0 \leq N\delta$ with N satisfying (5.25), thanks to

$$m(\xi_0^{-1})\xi_0 \leq m((N\delta)^{-1})N\delta \leq N^{1-\alpha+\sigma}m(\delta^{-1})\delta \leq \frac{4}{\alpha-\sigma}m(\delta^{-1})\delta, \quad (5.30)$$

and using (5.28) again, the positive contribution which is treated by (5.20) and (5.22) can further be bounded by

$$\begin{aligned} & -\partial_{\xi_0}\omega(\xi, \xi_0)\dot{\xi}_0 + \Omega_{x,e}(\xi, t)\partial_{\xi}\omega(\xi, \xi_0) \\ & \leq -\frac{1}{2}D_{x,e}(\xi, t) + \kappa(m(\delta^{-1}))^2\delta \left(\frac{4\rho\alpha}{\alpha-\sigma}\frac{\gamma}{\kappa} + C_2(C_0+2)\beta^2\kappa + \frac{4C_2(C_0+1)\beta}{(\alpha-\sigma)^2}\gamma \right). \end{aligned}$$

For the negative contribution from the diffusion term, from (5.18) and (5.24), we directly get that by letting $\gamma \leq \frac{(1-\beta)(\alpha-\sigma)}{8}\kappa$,

$$\begin{aligned} D_{x,e}(\xi, t) &\leq -\frac{2C_1}{\alpha}m(\xi^{-1})((1-\beta)\kappa m(\delta^{-1})\delta - \gamma m(\xi_0^{-1})\xi_0) \\ &\leq -\frac{2C_1}{\alpha} \left((1-\beta)\kappa - \frac{4\gamma}{\alpha-\sigma} \right) (m(\delta^{-1})^2\delta) \\ &\leq -\frac{C_1(1-\beta)\kappa}{\alpha} (m(\delta^{-1})^2\delta). \end{aligned} \quad (5.31)$$

Hence for $\xi_0 \leq N\delta$, we have

$$\text{L.H.S. of (5.14)} \leq \kappa(m(\delta^{-1}))^2\delta \left(\frac{4\alpha}{\alpha-\sigma}\rho + C_2(C_0+2)\beta^2\kappa + \frac{4C_2(C_0+1)\beta}{(\alpha-\sigma)^2}\gamma - \frac{C_1(1-\beta)}{2\alpha} \right) < 0,$$

where the last inequality is ensured if we set

$$\begin{aligned} \rho &< \frac{C_1(1-\beta)(\alpha-\sigma)}{24\alpha^2}, \quad \kappa < \min \left\{ \frac{C_1(1-\beta)}{6C_2(C_0+2)\beta^2\alpha}, \frac{1}{4C_2\beta} \right\}, \\ \gamma &\leq \min \left\{ \frac{(1-\beta)(\alpha-\sigma)}{8}\kappa, \frac{C_1(1-\beta)(\alpha-\sigma)^2}{24C_2(C_0+1)\beta\alpha} \right\}. \end{aligned} \quad (5.32)$$

Case 2: $\xi_0 > \delta, \delta < \xi \leq \xi_0$.

From $\omega(\xi, \xi_0) = \kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi_0} m(\eta^{-1})d\eta - \gamma m(\xi_0^{-1})\xi_0 + \gamma m(\xi_0^{-1})\xi$ in this case, we have

$$\partial_{\xi_0}\omega(\xi, \xi_0) = \gamma m'(\xi_0^{-1})\xi_0^{-2}(\xi_0 - \xi) \leq \alpha\gamma m(\xi_0^{-1}), \quad \text{and} \quad \partial_{\xi}\omega(\xi, \xi_0) = \gamma m(\xi_0^{-1}),$$

and

$$\begin{aligned} \omega(\xi, \xi_0) &\geq \omega(\delta, \xi_0) \geq \kappa m(\delta^{-1})\delta + \gamma \xi_0^{\alpha-\sigma} m(\xi_0^{-1}) \frac{1}{1-\alpha+\sigma} (\xi_0^{1-\alpha+\sigma} - \delta^{1-\alpha+\sigma}) - \gamma m(\xi_0^{-1})(\xi_0 - \delta) \\ &\geq \kappa m(\delta^{-1})\delta + \frac{\gamma}{1-\alpha+\sigma} m(\xi_0^{-1}) \xi_0^{\alpha-\sigma} (\xi_0^{1-\alpha+\sigma} - \delta^{1-\alpha+\sigma}) - \gamma m(\xi_0^{-1})(\xi_0 - \delta) \\ &= M_{\xi_0, \delta} + \beta \kappa m(\delta^{-1})\delta, \end{aligned}$$

and

$$\omega(\xi, \xi_0) - \omega(0+, \xi_0) \leq \omega(\xi_0, \xi_0) - \omega(0+, \xi_0) = \gamma m(\xi_0^{-1})(\xi_0 - \delta) + \beta \kappa m(\delta^{-1})\delta. \quad (5.33)$$

Thus by using (5.5), we get

$$-\partial_{\xi_0} \omega(\xi, \xi_0) \dot{\xi}_0(t) \leq \alpha \rho \gamma \left(m(\xi_0^{-1}) \right)^2 \xi_0. \quad (5.34)$$

From the following estimate

$$\begin{aligned} \int_{\xi}^{\infty} \frac{\omega(\eta, \xi_0)}{\eta^2} d\eta &= \frac{\omega(\xi, \xi_0)}{\xi} + \int_{\xi}^{\xi_0} \frac{\gamma m(\xi_0^{-1})}{\eta} d\eta + \int_{\xi_0}^{\infty} \frac{\gamma m(\eta^{-1})}{\eta} d\eta \\ &\leq \frac{\omega(\xi, \xi_0)}{\xi} + \gamma m(\xi_0^{-1}) \left(\log \frac{\xi_0}{\xi} \right) + \gamma \xi_0^{\alpha-\sigma} m(\xi_0^{-1}) \int_{\xi_0}^{\infty} \frac{1}{\eta^{1-\alpha+\sigma}} d\eta \\ &= \frac{\omega(\xi, \xi_0)}{\xi} + \gamma m(\xi_0^{-1}) \left(\log \frac{\xi_0}{\xi} \right) + \frac{\gamma}{\alpha - \sigma} m(\xi_0^{-1}), \end{aligned}$$

and similarly as obtaining (5.22), we find that for $\gamma \leq \frac{1}{4C_2}$,

$$\begin{aligned} &\Omega_{x,e}(\xi, t) \partial_{\xi} \omega(\xi, \xi_0) \\ &\leq -C_2 \gamma D_{x,e}(\xi, t) + 2C_2 \gamma \omega(\xi, \xi_0) m(\xi_0^{-1}) + C_2 \left(\gamma m(\xi_0^{-1}) \right)^2 \left(\xi \log \frac{\xi_0}{\xi} + \frac{\xi}{\alpha - \sigma} \right) \\ &\leq -\frac{1}{4} D(\xi, t) + \frac{C_2(C_0 + 3)}{\alpha - \sigma} \left(\gamma m(\xi_0^{-1}) \right)^2 \xi_0 + 2C_2 \beta \gamma \kappa m(\delta^{-1}) \delta m(\xi_0^{-1}) + 2C_2 \gamma m(\xi_0^{-1}) \omega(0+, \xi_0), \end{aligned} \quad (5.35)$$

where $C_0 > 0$ is the constant such that $\frac{\xi}{\xi_0} \log \frac{\xi_0}{\delta} \leq C_0$. For the contribution from the diffusion term, we also have (5.23) and (5.24). If $\xi_0 \geq N\delta$ with $N \in \mathbb{N}$ defined by (5.25), by using (5.26) and setting $\gamma < \frac{C_1}{4C_2\alpha}$, we deduce that

$$\begin{aligned} \text{L.H.S. of (5.14)} &\leq \kappa m(\delta^{-1}) \delta m(\xi^{-1}) \left(2C_2 \beta \gamma - \frac{C_1(1-\beta)}{\alpha} \right) + \\ &\quad + \gamma m(\xi_0^{-1}) \xi_0 m(\xi^{-1}) \left(\rho \alpha + \frac{C_2(C_0 + 3)}{\alpha - \sigma} \gamma - \frac{C_1(\alpha - \sigma)}{\alpha} \right) < 0, \end{aligned}$$

where the last inequality is guaranteed as long as

$$\rho < \frac{C_1(\alpha - \sigma)}{2\alpha^2}, \quad \gamma < \min \left\{ \frac{1}{4C_2}, \frac{C_1}{4C_2\alpha}, \frac{C_1(1-\beta)}{2C_2\beta\alpha}, \frac{C_1(\alpha - \sigma)^2}{2C_2(C_0 + 3)\alpha} \right\}. \quad (5.36)$$

If $\xi_0 \leq N\delta$ with N satisfying (5.25), by applying (5.30) and setting $\gamma < \frac{C_1}{4C_2\alpha}$, the positive contribution treated by (5.34) and (5.35) can further be bounded as

$$\begin{aligned} &-\partial_{\xi_0} \omega(\xi, \xi_0) \dot{\xi}_0(t) + \Omega_{x,e}(\xi, t) \partial_{\xi} \omega(\xi, \xi_0) \\ &\leq -\frac{1}{2} D_{x,e}(\xi, t) + \frac{4\gamma}{\alpha - \sigma} m(\delta^{-1}) \delta m(\xi_0^{-1}) \left(\rho \alpha + \frac{C_2(C_0 + 3)}{\alpha - \sigma} \gamma \right). \end{aligned}$$

For the negative contribution from the diffusion term, by arguing as (5.31) we obtain that for $\gamma \leq \frac{(1-\beta)(\alpha-\sigma)}{8} \kappa$,

$$\begin{aligned} D_{x,e}(\xi, t) &\leq -\frac{2C_1}{\alpha} m(\xi^{-1}) \left((1-\beta) \kappa m(\delta^{-1}) \delta - \frac{4\gamma}{\alpha - \sigma} m(\delta^{-1}) \delta \right) \\ &\leq -\frac{2C_1}{\alpha} \frac{4\gamma}{\alpha - \sigma} m(\xi^{-1}) m(\delta^{-1}) \delta. \end{aligned}$$

Hence for $\xi_0 \leq N\delta$ with N given by (5.25), we have

$$\text{L.H.S. of (5.14)} \leq \frac{4\gamma}{\alpha - \sigma} m(\delta^{-1}) \delta m(\xi^{-1}) \left(\rho\alpha + \frac{C_2(C_0 + 3)}{\alpha - \sigma} \gamma - \frac{C_1}{\alpha} \right) < 0,$$

where the last inequality is ensured if we set

$$\rho < \frac{C_1(\alpha - \sigma)}{2\alpha^2}, \quad \gamma \leq \min \left\{ \frac{(1 - \beta)(\alpha - \sigma)}{8} \kappa, \frac{1}{4C_2}, \frac{C_1(\alpha - \sigma)}{2C_2(C_0 + 3)\alpha} \right\}. \quad (5.37)$$

Case 3: $\xi_0 > \delta$, $\xi > \xi_0$.

In this case, from $\omega(\xi, \xi_0) = \kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi} m(\eta^{-1}) d\eta$, we see that $\partial_{\xi_0} \omega(\xi, \xi_0) = 0$, $\partial_{\xi} \omega(\xi, \xi_0) = \gamma m(\xi^{-1})$, and

$$\begin{aligned} \int_{\xi}^{\infty} \frac{\omega(\eta, \xi_0)}{\eta^2} d\eta &= \frac{\omega(\xi, \xi_0)}{\xi} + \int_{\xi}^{\infty} \frac{\gamma m(\eta^{-1})}{\eta} d\eta \\ &\leq \frac{\omega(\xi, \xi_0)}{\xi} + \gamma \xi^{\alpha - \sigma} m(\xi^{-1}) \int_{\xi}^{\infty} \frac{1}{\eta^{1 + \alpha - \sigma}} d\eta \leq \frac{\omega(\xi, \xi_0)}{\xi} + \frac{\gamma}{\alpha - \sigma} m(\xi^{-1}). \end{aligned}$$

Thus thanks to (5.16), we get

$$\Omega_{x,e}(\xi, t) \partial_{\xi} \omega(\xi, \xi_0) \leq -C_2 \gamma D_{x,e}(\xi, t) + C_2 \left(2\omega(\xi, \xi_0) + \frac{\gamma}{\alpha - \sigma} \xi m(\xi^{-1}) \right) \gamma m(\xi^{-1}). \quad (5.38)$$

For the contribution from the diffusion term, since $\omega(2\eta + \xi, \xi_0) - \omega(2\eta - \xi, \xi_0) \leq \omega(2\xi, \xi_0) < 2\omega(\xi, \xi_0)$, by estimating as (5.23) we obtain

$$D_{x,e}(x, t) \leq C_1 (\omega(2\xi, \xi_0) - 2\omega(\xi, \xi_0)) \int_{\frac{\xi}{2}}^{\infty} \frac{m(\eta^{-1})}{\eta} d\eta \leq \frac{C_1}{\alpha} (\omega(2\xi, \xi_0) - 2\omega(\xi, \xi_0)) m(\xi^{-1}). \quad (5.39)$$

Observing that

$$\omega(2\xi, \xi_0) - \omega(\xi, \xi_0) = \gamma \int_{\xi}^{2\xi} m(\eta^{-1}) d\eta \leq \gamma \xi^{\alpha - \sigma} m(\xi^{-1}) \int_{\xi}^{2\xi} \frac{1}{\eta^{\alpha - \sigma}} d\eta \leq \frac{2^{1 - \alpha + \sigma} - 1}{1 - \alpha + \sigma} \gamma m(\xi^{-1}) \xi,$$

and

$$\omega(\xi, \xi_0) \geq \gamma \int_{\delta}^{\xi} m(\eta^{-1}) d\eta \geq \gamma \xi^{\alpha - \sigma} m(\xi^{-1}) \int_{\delta}^{\xi} \frac{1}{\eta^{\alpha - \sigma}} d\eta \geq \gamma \xi^{\alpha - \sigma} m(\xi^{-1}) \frac{\xi^{1 - \alpha + \sigma} - \delta^{1 - \alpha + \sigma}}{1 - \alpha + \sigma},$$

thus if ξ satisfies that $\xi \geq \delta \left(\frac{1}{2^{\alpha - \sigma} - 1} \right)^{\frac{1}{1 - \alpha + \sigma}}$, equivalently, $\xi^{1 - \alpha + \sigma} - \delta^{1 - \alpha + \sigma} \geq (2 - 2^{\alpha - \sigma}) \xi^{1 - \alpha + \sigma}$, we find

$$\omega(\xi, \xi_0) \geq \frac{2 - 2^{\alpha - \sigma}}{1 - \alpha + \sigma} \gamma m(\xi^{-1}) \xi = 2^{\alpha - \sigma} \frac{2^{1 - (\alpha - \sigma)} - 1}{1 - (\alpha - \sigma)} \gamma m(\xi^{-1}) \xi \geq \tilde{c} \gamma m(\xi^{-1}) \xi, \quad (5.40)$$

and

$$\omega(2\xi, \xi_0) - \omega(\xi, \xi_0) \leq \frac{2^{1 - \alpha + \sigma} - 1}{2 - 2^{\alpha - \sigma}} \omega(\xi, \xi_0) = 2^{-\alpha + \sigma} \omega(\xi, \xi_0),$$

and

$$\omega(2\xi, \xi_0) - 2\omega(\xi, \xi_0) \leq -(1 - 2^{-\alpha + \sigma}) \omega(\xi, \xi_0) \leq -\frac{\tilde{c}(\alpha - \sigma)}{2} \omega(\xi, \xi_0), \quad (5.41)$$

with \tilde{c} defined by (4.20). Hence if $\xi \geq \delta \left(\frac{1}{2^{\alpha - \sigma} - 1} \right)^{\frac{1}{1 - \alpha + \sigma}}$, and by gathering the above estimates and setting $\gamma \leq \frac{1}{2C_2}$, we deduce that

$$\Omega_{x,e}(\xi, t) \partial_{\xi} \omega(\xi, \xi_0) \leq -\frac{1}{2} D_{x,e}(\xi, t) + \frac{3C_2}{\tilde{c}(\alpha - \sigma)} \gamma \omega(\xi, \xi_0) m(\xi^{-1}),$$

and

$$D_{x,e}(\xi, t) \leq -\frac{C_1 \tilde{c}(\alpha - \sigma)}{2\alpha} \omega(\xi, \xi_0) m(\xi^{-1}),$$

and

$$\Omega_{x,e}(\xi, t) \partial_\xi \omega(\xi, \xi_0) + D_{x,e}(\xi, t) \leq \left(\frac{3C_2}{\tilde{c}(\alpha - \sigma)} \gamma - \frac{C_1 \tilde{c}(\alpha - \sigma)}{2\alpha} \right) \omega(\xi, \xi_0) m(\xi^{-1}) < 0,$$

where the last inequality is ensured if we set

$$\gamma < \min \left\{ \frac{1}{2C_2}, \frac{C_1 \tilde{c}^2(\alpha - \sigma)^2}{6C_2 \alpha} \right\}. \quad (5.42)$$

On the other hand, if ξ satisfies that $\xi \leq \delta \left(\frac{1}{2^{\alpha-\sigma}-1} \right)^{\frac{1}{1-\alpha+\sigma}}$, since $\omega(\xi, \xi_0) - \omega(0+, \xi_0)$ is concave and $\omega(0+, \xi_0) \geq (1 - \beta) \kappa m(\delta^{-1}) \delta$, we get

$$D_{x,e}(\xi, t) \leq -2\omega(0+, \xi_0) \int_{\frac{\xi}{2}}^{\infty} \frac{m(\eta^{-1})}{\eta} d\eta \leq -\frac{2(1-\beta)\kappa}{\alpha} \delta m(\delta^{-1}) m(\xi^{-1}), \quad (5.43)$$

and by using $\xi^{1-\alpha+\sigma} - \delta^{1-\alpha+\sigma} \leq \delta^{1-\alpha+\sigma} \frac{2-2^{\alpha-\sigma}}{2^{\alpha-\sigma}-1}$, we also infer that

$$m(\xi^{-1}) \xi \leq \delta^{\alpha-\sigma} m(\delta^{-1}) \xi^{1-\alpha+\sigma} \leq m(\delta^{-1}) \delta \frac{1}{2^{\alpha-\sigma}-1} \leq \frac{1}{\tilde{c}(\alpha - \sigma)} m(\delta^{-1}) \delta,$$

and

$$\begin{aligned} \omega(\xi, \xi_0) &\leq \kappa m(\delta^{-1}) \delta + \gamma \delta^{\alpha-\sigma} m(\delta^{-1}) \int_{\delta}^{\xi} \frac{1}{\eta^{\alpha-\sigma}} d\eta \\ &\leq \kappa m(\delta^{-1}) \delta + \frac{\gamma}{1-\alpha+\sigma} \delta^{\alpha-\sigma} m(\delta^{-1}) (\xi^{1-\alpha+\sigma} - \delta^{1-\alpha+\sigma}) \\ &\leq \left(\kappa + \frac{2^{\alpha-\sigma}(2^{1-\alpha+\sigma}-1)}{1-\alpha+\sigma} \frac{1}{2^{\alpha-\sigma}-1} \gamma \right) m(\delta^{-1}) \delta \leq \left(\kappa + \frac{2\gamma}{\tilde{c}(\alpha - \sigma)} \right) m(\delta^{-1}) \delta, \end{aligned} \quad (5.44)$$

where \tilde{c} is given by (4.20) and we also used $\sup_{x \in [0,1]} \frac{2^x-1}{x} \leq 1$. Hence if $\xi \leq \delta \left(\frac{1}{2^{\alpha-\sigma}-1} \right)^{\frac{1}{1-\alpha+\sigma}}$, by collecting the above results and letting $\gamma \leq \min \left\{ \frac{1}{2C_2}, \kappa \right\}$, we obtain

$$\Omega_{x,e}(\xi, t) \partial_\xi \omega(\xi, \xi_0) \leq -\frac{1}{2} D_{x,e}(\xi, t) + \left(2\gamma + \frac{4\gamma}{\tilde{c}(\alpha - \sigma)} + \frac{2\gamma}{\tilde{c}(\alpha - \sigma)^2} \right) \kappa \delta m(\delta^{-1}) m(\xi^{-1}),$$

and thus

$$\Omega_{x,e}(\xi, t) \partial_\xi \omega(\xi, \xi_0) + D_{x,e}(\xi, t) \leq \left(\frac{8\gamma}{\tilde{c}(\alpha - \sigma)^2} - \frac{1-\beta}{\alpha} \right) \kappa \delta m(\delta^{-1}) m(\xi^{-1}),$$

where the last inequality is ensured by setting

$$\gamma < \min \left\{ \frac{\tilde{c}(1-\beta)(\alpha - \sigma)^2}{8\alpha}, \frac{1}{2C_2}, \kappa \right\}. \quad (5.45)$$

Case 4: $0 < \xi_0 \leq \delta$, $0 < \xi \leq \xi_0$.

In this case $\omega(\xi, \xi_0) = (1 - \beta) \kappa m(\delta^{-1}) \delta^{1-\beta} \xi_0^\beta + \beta \kappa m(\delta^{-1}) \delta^{1-\beta} \xi_0^{\beta-1} \xi$, and thus

$$\partial_{\xi_0} \omega(\xi, \xi_0) = \beta(1 - \beta) \kappa m(\delta^{-1}) \left(\frac{\delta}{\xi_0} \right)^{1-\beta} \left(1 - \frac{\xi}{\xi_0} \right), \quad \text{and} \quad \partial_\xi \omega(\xi, \xi_0) = \beta \kappa m(\delta^{-1}) \left(\frac{\delta}{\xi_0} \right)^{1-\beta},$$

and

$$\begin{aligned} \omega(\xi, \xi_0) &\geq \omega(0+, \xi_0) \geq (1 - \beta) \kappa m(\delta^{-1}) \delta^{1-\beta} \xi_0^\beta, \\ \omega(\xi, \xi_0) &\leq \omega(\delta, \xi_0) \leq \kappa m(\delta^{-1}) \delta^{1-\beta} \xi_0^\beta. \end{aligned} \quad (5.46)$$

Taking advantage of the following estimates

$$m(\delta^{-1}) \leq \left(\frac{\xi}{\delta}\right)^{\alpha-\sigma} m(\xi^{-1}), \quad \text{and} \quad m(\xi_0^{-1}) \leq \left(\frac{\xi}{\xi_0}\right)^{\alpha-\sigma} m(\xi^{-1}), \quad (5.47)$$

we deduce

$$\begin{aligned} -\partial_{\xi_0} \omega(\xi, \xi_0) \dot{\xi}_0(t) &\leq \rho\beta(1-\beta)\kappa m(\delta^{-1}) \left(\frac{\delta}{\xi_0}\right)^{1-\beta} (\xi_0 - \xi) m(\xi_0^{-1}) \\ &\leq \rho\beta(1-\beta)\kappa m(\delta^{-1}) \left(\frac{\delta}{\xi_0}\right)^{1-\beta} \xi_0 \left(\frac{\xi}{\xi_0}\right)^{\alpha-\sigma} m(\xi^{-1}) \\ &\leq \rho\beta(1-\beta)\kappa m(\delta^{-1}) \xi_0 \left(\frac{\delta}{\xi_0}\right)^{1+\alpha-\sigma-\beta} \left(\frac{\xi}{\delta}\right)^{\alpha-\sigma} m(\xi^{-1}) \\ &\leq \rho\beta(1-\beta)\kappa m(\delta^{-1}) \xi_0 m(\xi^{-1}), \end{aligned} \quad (5.48)$$

and

$$-\frac{C_2}{m(\xi^{-1})} D_{x,e}(\xi, t) \partial_{\xi} \omega(\xi, \xi_0) \leq -C_2 \beta \kappa \left(\frac{\delta}{\xi_0}\right)^{1-\beta-\alpha+\sigma} D_{x,e}(\xi, t) \leq -C_2 \beta \kappa D_{x,e}(\xi, t).$$

In view of the integration by parts and (5.2), we see that

$$\begin{aligned} \int_{\xi}^{\infty} \frac{\omega(\eta, \xi_0)}{\eta^2} d\eta &= \frac{\omega(\xi, \xi_0)}{\xi} + \int_{\xi}^{\infty} \frac{\partial_{\eta} \omega(\eta, \xi_0)}{\eta} d\eta \\ &\leq \frac{\omega(\xi, \xi_0)}{\xi} + \int_{\xi}^{\xi_0} \frac{\beta \kappa m(\delta^{-1}) \delta^{1-\beta} \xi_0^{\beta-1}}{\eta} d\eta + \int_{\xi_0}^{\infty} \frac{\beta \kappa m(\delta^{-1}) \delta^{1-\beta} \eta^{\beta-1}}{\eta} d\eta \\ &\leq \frac{\omega(\xi, \xi_0)}{\xi} + \beta \kappa m(\delta^{-1}) \left(\frac{\delta}{\xi_0}\right)^{1-\beta} \left(\log \frac{\xi_0}{\xi}\right) + \frac{\beta}{1-\beta} \kappa m(\delta^{-1}) \left(\frac{\delta}{\xi_0}\right)^{1-\beta}, \end{aligned}$$

then gathering the above estimates and (5.16) leads to that for $\kappa \leq \frac{1}{2C_2\beta}$,

$$\begin{aligned} \Omega_{x,e}(\xi, t) \partial_{\xi} \omega(\xi, \xi_0) &\leq -C_2 \beta \kappa D_{x,e}(\xi, t) + 2C_2 \beta \kappa m(\delta^{-1}) \left(\frac{\delta}{\xi_0}\right)^{1-\beta} \omega(\xi, \xi_0) \\ &\quad + C_2 (\beta \kappa m(\delta^{-1}))^2 \left(\frac{\delta}{\xi_0}\right)^{2(1-\beta)} \xi_0 \left(\frac{\xi}{\xi_0} \log \frac{\xi_0}{\xi} + \frac{\xi}{\xi_0} \frac{1}{1-\beta}\right) \\ &\leq -\frac{1}{2} D_{x,e}(\xi, t) + C_2 \beta (\kappa m(\delta^{-1}))^2 \left(\frac{\delta}{\xi_0}\right)^{2(1-\beta)} \xi_0 \left(2 + C_0 \beta + \frac{\beta}{1-\beta}\right), \end{aligned} \quad (5.49)$$

where in the third line we used $\frac{\xi}{\xi_0} \leq 1$ and $\frac{\xi}{\xi_0} \left(\log \frac{\xi_0}{\xi}\right) \leq C_0$. For the contribution from the diffusion term, by arguing as (5.24), we obtain

$$D_{x,e}(\xi, t) \leq -2C_1 \omega(0+, \xi_0) \int_{\frac{\xi}{2}}^{\infty} \frac{m(\eta^{-1})}{\eta} d\eta \leq -\frac{2(1-\beta)C_1}{\alpha} \kappa m(\delta^{-1}) \left(\frac{\delta}{\xi_0}\right)^{1-\beta} \xi_0 m(\xi^{-1}). \quad (5.50)$$

Collecting the estimates (5.48), (5.49) and (5.50), and using (5.47) again, we find that

L.H.S. of (5.14)

$$\begin{aligned} &\leq \kappa m(\delta^{-1}) \left(\frac{\delta}{\xi_0}\right)^{1-\beta} \xi_0 m(\xi^{-1}) \left(\rho\beta(1-\beta) + \frac{C_2\beta(2+C_0\beta)\kappa}{1-\beta} \left(\frac{\xi}{\xi_0}\right)^{\alpha-\sigma} \left(\frac{\delta}{\xi_0}\right)^{1-\beta-\alpha+\sigma} - \frac{C_1(1-\beta)}{\alpha} \right) \\ &\leq \kappa m(\delta^{-1}) \left(\frac{\delta}{\xi_0}\right)^{1-\beta} \xi_0 m(\xi^{-1}) \left(\rho\beta(1-\beta) + \frac{C_2\beta(2+C_0\beta)\kappa}{1-\beta} - \frac{C_1(1-\beta)}{\alpha} \right), \end{aligned}$$

which leads to the desired inequality (5.14) as long as ρ, κ are such that

$$\rho < \frac{C_1}{2\alpha\beta}, \quad \kappa < \min \left\{ \frac{1}{2C_2\beta}, \frac{C_1(1-\beta)^2}{2C_2(2+C_0\beta)\beta\alpha} \right\}. \quad (5.51)$$

Case 5: $0 < \xi_0 \leq \delta$, $\xi_0 < \xi \leq \delta$.

Similarly as obtaining (4.14), we have that by setting $\kappa \leq \frac{1}{2C_2}$ and $\gamma \leq \kappa$,

$$\Omega_{x,e}(\xi, t) \partial_\xi \omega(\xi, \xi_0) \leq -\frac{1}{2} D_{x,e}(\xi, t) + \frac{3C_2}{1-\beta} \beta \left(\kappa m(\delta^{-1}) \delta^{1-\beta} \right)^2 \xi^{2\beta-1}. \quad (5.52)$$

For the contribution from the diffusion term, if $\xi_0 \leq \frac{\xi}{4}$, then by arguing as (4.16) we find

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C_1 \omega''(\xi) \int_0^{\frac{\xi}{2}} \eta m(\eta^{-1}) d\eta \\ &\leq -C_1 \beta (1-\beta) \kappa m(\delta^{-1}) \delta^{1-\beta} \xi^{\beta-2} \int_{\xi_0}^{\frac{\xi}{2}} \eta m(\eta^{-1}) d\eta \\ &\leq -C_1 \beta (1-\beta) \kappa (m(\delta^{-1}))^2 \delta^{1+\alpha-\sigma-\beta} \xi^{\beta-2} \int_{\frac{\xi}{4}}^{\frac{\xi}{2}} \eta^{1-\alpha+\sigma} d\eta \\ &\leq -C_1 \beta (1-\beta) \kappa (m(\delta^{-1}))^2 \delta^{1+\alpha-\sigma-\beta} \xi^{\beta-2} \frac{1}{2-\alpha+\sigma} \left(\left(\frac{\xi}{2} \right)^{2-\alpha+\sigma} - \left(\frac{\xi}{4} \right)^{2-\alpha+\sigma} \right) \\ &\leq -\frac{C_1 \beta (1-\beta) \kappa}{32} (m(\delta^{-1}))^2 \delta^{1+\alpha-\sigma-\beta} \xi^{\beta-\alpha+\sigma}. \end{aligned} \quad (5.53)$$

Thus for $\xi_0 \leq \frac{\xi}{4}$, we get that by letting $\kappa < \frac{C_1(1-\beta)^2}{192C_2}$,

$$\begin{aligned} \Omega_{x,e}(\xi, t) \partial_\xi \omega(\xi, \xi_0) + D_{x,e}(\xi, t) &\leq \beta \kappa (m(\delta^{-1}))^2 \delta^{1+\alpha-\sigma-\beta} \xi^{\beta-\alpha+\sigma} \left(\frac{3C_2}{1-\beta} \left(\frac{\delta}{\xi} \right)^{1-\alpha+\sigma-\beta} \kappa - \frac{C_1(1-\beta)}{64} \right) \\ &\leq \beta \kappa (m(\delta^{-1}))^2 \delta^{1+\alpha-\sigma-\beta} \xi^{\beta-\alpha+\sigma} \left(\frac{3C_2}{1-\beta} \kappa - \frac{C_1(1-\beta)}{64} \right) < 0. \end{aligned}$$

Whereas if $\xi_0 \geq \frac{\xi}{4}$, by using (5.15), the concavity property of $\omega(\eta, \xi_0) - \omega(0+, \xi_0)$ for $\eta \geq 0$ and (5.47), we get

$$\begin{aligned} D_{x,e}(\xi, t) &\leq -C_1 2\omega(0+, \xi_0) \int_{\frac{\xi}{2}}^{\infty} \frac{m(\eta^{-1})}{\eta} d\eta \\ &\leq -\frac{2C_1(1-\beta)\kappa m(\delta^{-1}) \delta^{1-\beta} \xi_0^\beta}{\alpha} m(\xi^{-1}) \\ &\leq -\frac{C_1(1-\beta)\kappa m(\delta^{-1}) \delta^{1-\beta}}{2\alpha} \xi^\beta m(\xi^{-1}) \leq -\frac{C_1(1-\beta)\kappa}{2\alpha} (m(\delta^{-1}))^2 \delta^{1+\alpha-\sigma-\beta} \xi^{\beta-\alpha+\sigma}. \end{aligned} \quad (5.54)$$

Thus combining this estimate with (5.52) yields that

$$\Omega_{x,e}(\xi, t) \partial_\xi \omega(\xi, \xi_0) + D_{x,e}(\xi, t) \leq \kappa (m(\delta^{-1}))^2 \delta^{1+\alpha-\sigma-\beta} \xi^{\beta-\alpha+\sigma} \left(\frac{3C_2\beta}{1-\beta} \kappa - \frac{C_1(1-\beta)}{2\alpha} \right) < 0,$$

where the last inequality is ensured by setting $\kappa < \frac{C_1(1-\beta)^2}{6C_2\beta}$. Notice that in this case the conditions on κ and γ are

$$\kappa < \min \left\{ \frac{1}{2C_2}, \frac{C_1(1-\beta)^2}{192C_2} \right\}, \quad \gamma \leq \kappa. \quad (5.55)$$

Case 6: $0 < \xi_0 \leq \delta$, $\xi > \delta$.

The proof of this case is almost the same as the proof of Case 2 in Section 4, and we omit the details. Note that the conditions on κ, γ are given by (5.9).

Therefore, for the MOC $\omega(\xi, \xi_0)$ defined by (5.1)-(5.2) and $\xi_0 = \xi_0(t)$ defined by (5.5) with ρ, κ, γ are appropriate constants satisfying (5.9), (5.29), (5.32), (5.36), (5.37), (5.42), (5.51), (5.55), we justify (5.14) for all $\xi > 0$ and $t > 0$ based on the above analysis, and thus conclude Lemma 5.1. Observing that by suppressing the dependence on the constants C_1, C_2, \tilde{c} and C_0 , the conditions on ρ, κ, γ are as follows

$$\rho \leq \frac{1}{C} \frac{(1-\beta)(\alpha-\sigma)}{\alpha^2}, \quad \kappa \leq \frac{1}{C}(1-\beta)^2, \quad \gamma \leq \frac{1}{C} \min \{ \beta(1-\beta)^2, (1-\beta)^3(\alpha-\sigma) \}, \quad (5.56)$$

with $C > 0$ some constant independent of α, σ, β .

□

6. PROOF OF THEOREMS 1.4, 1.5 AND 1.7

6.1. Proof of Theorem 1.4: eventual regularity of vanishing viscosity solution. Consider the following approximate system

$$\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon + \mathcal{L} \theta^\epsilon - \epsilon \Delta \theta^\epsilon = 0, \quad u^\epsilon = \mathcal{P}(\theta^\epsilon), \quad \theta^\epsilon|_{t=0} = \theta_0^\epsilon = \phi_\epsilon * (\theta_0 1_{B_{1/\epsilon}}), \quad (6.1)$$

where $\operatorname{div} u^\epsilon = 0$, \mathcal{P} is composed of zero-order pseudo-differential operators defined by (1.2), $1_{B_{1/\epsilon}}$ is the indicator function on the ball $B_{1/\epsilon}$, $\phi_\epsilon(x) = \epsilon^{-d} \phi(\epsilon^{-1}x)$, and $\phi \in C_c^\infty(\mathbb{R}^d)$ is a radial test function satisfying $\int_{\mathbb{R}^d} \phi = 1$. Since $\theta_0 \in L^2(\mathbb{R}^d)$, we have $\|\theta_0^\epsilon\|_{L^2} \leq \|\theta_0\|_{L^2}$, and $\|\theta_0^\epsilon\|_{H^s} \lesssim_{\epsilon, s} \|\theta_0\|_{L^2}$ for every $s > 0$. For $\epsilon > 0$ and $s > d/2 + 1$, we have the following lemma concerning the global well-posedness for the approximate system (6.1).

Lemma 6.1. *For every $\epsilon > 0$, the Cauchy problem of the approximate drift-diffusion equation (6.1) admits a uniquely global smooth solution $\theta^\epsilon(x, t)$ such that*

$$\theta^\epsilon \in C([0, \infty[; H^s(\mathbb{R}^d)) \cap C^\infty([0, \infty[\times \mathbb{R}^d), \quad \text{with } s > d/2 + 1.$$

The proof of this lemma is more or less standard, and one can refer to [29, Theorem 1.4] (at $\alpha = 2$ case) for the use of the nonlocal maximum principle method, and we omit the details here.

Since u^ϵ is divergence-free, we can also show the uniform-in- ϵ energy estimate. By taking the L^2 -inner product of the equation (6.1) with θ^ϵ , and using the integration by parts, we have

$$\frac{1}{2} \frac{d}{dt} \|\theta^\epsilon(t)\|_{L^2}^2 + \int_{\mathbb{R}^d} \mathcal{L}(\theta^\epsilon)(x, t) \theta^\epsilon(x, t) dx + \epsilon \|\nabla \theta^\epsilon(t)\|_{L^2}^2 = 0. \quad (6.2)$$

Since the symbol of \mathcal{L} satisfies $A(\zeta) \geq 0$ from (1.13) and (1.21), we see that

$$\int_{\mathbb{R}^d} (\mathcal{L} \theta^\epsilon)(x, t) \theta^\epsilon(x, t) dx = \int_{\mathbb{R}^d} A(\zeta) |\hat{\theta}^\epsilon(\zeta, t)|^2 d\zeta \geq 0. \quad (6.3)$$

Inserting (6.3) into (6.2) leads to $\frac{d}{dt} \|\theta^\epsilon(t)\|_{L^2}^2 \leq 0$, which by integrating in time implies

$$\|\theta^\epsilon(t)\|_{L^2} \leq \|\theta_0^\epsilon\|_{L^2} \leq \|\theta_0\|_{L^2}, \quad \forall t \geq 0. \quad (6.4)$$

By applying Lemma 2.3, we also obtain

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{L} \theta^\epsilon)(x, t) \theta^\epsilon(x, t) dx &\geq C^{-1} \int_{\mathbb{R}^d} |\zeta|^{\alpha-\sigma} |\hat{\theta}^\epsilon(\zeta, t)|^2 d\zeta - C \int_{\mathbb{R}^d} |\hat{\theta}^\epsilon(\zeta, t)|^2 d\zeta \\ &\geq C^{-1} \|\theta^\epsilon\|_{\dot{H}^{\frac{\alpha-\sigma}{2}}}^2 - C \|\theta^\epsilon\|_{L^2}^2. \end{aligned} \quad (6.5)$$

Plugging this estimate into (6.2), and using (6.4), we find

$$\frac{d}{dt} \|\theta^\epsilon(t)\|_{L^2}^2 + \frac{2}{C} \|\theta^\epsilon(t)\|_{\dot{H}^{\frac{\alpha-\sigma}{2}}}^2 \leq 2C \|\theta_0\|_{L^2}^2,$$

which ensures that for every $T > 0$,

$$\sup_{t \in [0, T]} \|\theta^\epsilon(t)\|_{L^2}^2 + \frac{2}{C} \int_0^T \|\theta^\epsilon(t)\|_{\dot{H}^{\frac{\alpha-\sigma}{2}}}^2 dt \leq (1 + 2CT) \|\theta_0\|_{L^2}^2. \quad (6.6)$$

Next based on the uniform L^2 estimate, we can use the De Giorgi's method to show the L^∞ -improvement, that is, for any fixed $t_0 > 0$ and every $T \geq t_0$, there is a constant $C_* > 0$ independent of ϵ and T so that

$$\sup_{t \in [t_0, T]} \|\theta^\epsilon(t)\|_{L^\infty} \leq C_* \left(\frac{1}{t_0} + C \right)^{\frac{d}{2(\alpha-\sigma)}} (1 + 2CT)^{\frac{1}{2}} \|\theta_0\|_{L^2}, \quad (6.7)$$

with $C > 0$ the constant appearing in (6.6). The proof is similar to that of [4, Corollary 4] or [13, Theorem 2.1], and here we sketch the main process in obtaining (6.7). Since the operator \mathcal{L} defined by (1.3) has nonnegative kernel K , by arguing as obtaining a corresponding inequality for fractional Laplacian operator in [14], we have that for every convex function ψ ,

$$\psi'(\theta^\epsilon) \mathcal{L}(\theta^\epsilon) \geq \mathcal{L}(\psi(\theta^\epsilon)).$$

We also find for every convex ψ , $-\psi'(\theta^\epsilon) \Delta \theta^\epsilon \geq -\Delta(\psi(\theta^\epsilon))$. For $M > 0$ chosen later (cf. (6.12)), applying the above two inequalities with

$$\psi(\theta^\epsilon) = (\theta^\epsilon - M_k)_+ =: \theta_k^\epsilon, \quad M_k := M(1 - 2^{-k}), \quad k \in \mathbb{N}, \quad (6.8)$$

we obtain the following pointwise inequality from (6.1),

$$\partial_t \theta_k^\epsilon + u^\epsilon \cdot \nabla \theta_k^\epsilon + \mathcal{L} \theta_k^\epsilon - \epsilon \Delta \theta_k^\epsilon \leq 0. \quad (6.9)$$

As deriving the energy estimate, we use (6.5) to get

$$\frac{1}{2} \frac{d}{dt} \|\theta_k^\epsilon(t)\|_{L^2}^2 + C^{-1} \|\theta_k^\epsilon(t)\|_{\dot{H}^{\frac{\alpha-\sigma}{2}}}^2 + \epsilon \|\nabla \theta_k^\epsilon(t)\|_{L^2}^2 \leq C \|\theta_k^\epsilon(t)\|_{L^2}^2. \quad (6.10)$$

Then for a fixed constant $t_0 > 0$ and every $T \geq t_0$, we denote $T_k := t_0(1 - 2^{-k})$, $k \in \mathbb{N}$, and the level set of energy as

$$U_k^\epsilon := \sup_{t \in [T_k, T]} \|\theta_k^\epsilon(t)\|_{L^2}^2 + \frac{2}{C} \int_{T_k}^T \|\theta_k^\epsilon(t)\|_{\dot{H}^{\frac{\alpha-\sigma}{2}}}^2 dt.$$

For some $s \in [T_{k-1}, T_k]$, we integrating (6.10) in time between s and $t \in [T_k, T]$, and also between s and T to find

$$\begin{aligned} \|\theta_k^\epsilon(t)\|_{L^2}^2 &\leq \|\theta_k^\epsilon(s)\|_{L^2}^2 + 2C \int_s^t \|\theta_k^\epsilon(t)\|_{L^2}^2 dt, \quad \text{and} \\ \frac{2}{C} \int_s^T \|\theta_k^\epsilon(t)\|_{\dot{H}^{\frac{\alpha-\sigma}{2}}}^2 dt &\leq \|\theta_k^\epsilon(s)\|_{L^2}^2 + 2C \int_s^T \|\theta_k^\epsilon(t)\|_{L^2}^2 dt, \end{aligned}$$

which implies $U_k^\epsilon \leq 2\|\theta_k^\epsilon(s)\|_{L^2}^2 + 4C \int_s^T \|\theta_k^\epsilon(t)\|_{\dot{H}^{\frac{\alpha-\sigma}{2}}}^2 dt$. Taking the mean in s on $[T_{k-1}, T_k]$, we infer

$$U_k^\epsilon \leq \left(\frac{2^{k+1}}{t_0} + 4C \right) \int_{T_{k-1}}^T \|\theta_k^\epsilon(t)\|_{L^2}^2 dt. \quad (6.11)$$

The inequality (6.11) is almost identical with (A.3) of [13], and we can proceed further to obtain

$$U_k^\epsilon \leq \left(\frac{2}{t_0} + 4C \right) \frac{2^{k(q-1)}}{M^{q-2}} (U_{k-1}^\epsilon)^{q/2}, \quad \text{with } q := 2 + \frac{2(\alpha - \sigma)}{d}.$$

Since $U_0^\epsilon \leq (1 + 2CT)\|\theta_0\|_{L^2}^2$, by choosing M (according to [37, Lemma 2.6]) to be

$$M = (1 + 2CT)^{1/2}\|\theta_0\|_{L^2} \left(2^{2+\frac{d}{\alpha-\sigma}} \left(\frac{2}{t_0} + 4C \right) \right)^{\frac{d}{2(\alpha-\sigma)}}, \quad (6.12)$$

we have $\lim_{k \rightarrow \infty} U_k^\epsilon = 0$, which ensures $\theta^\epsilon \leq M$ for all $t \in [t_0, T]$. The same result likewise holds for $-\theta^\epsilon$, and thus we conclude (6.7).

Hence, the uniform estimate (6.6) and (6.7) guarantee that, for some $t_0 > 0$ and every $T \geq t_0$, up to a subsequence θ^ϵ converges weakly (weakly-* in $L_t^\infty L_x^2$ and $L^\infty([t_0, T] \times \mathbb{R}^d)$) to some function θ belonging to

$$L^\infty([0, T]; L^2(\mathbb{R}^d)) \cap L^2([0, T]; \dot{H}^{\frac{\alpha-\sigma}{2}}(\mathbb{R}^d)) \cap L^\infty([t_0, T] \times \mathbb{R}^d). \quad (6.13)$$

Moreover, by using the compactness argument (e.g. [32, Proposition 6.3]), we can show that $\theta^\epsilon \rightarrow \theta$ and $u^\epsilon \rightarrow u = \mathcal{P}(\theta)$ both strongly in $L^2([0, T]; L_{\text{loc}}^2(\mathbb{R}^d))$. Thus we can pass the weak limit $\epsilon \rightarrow 0$ in the approximate system (6.1) to show that $\theta(x, t)$ is a global weak solution for the original equation (1.1)-(1.2), which satisfies the energy estimate (6.6) and L^∞ -estimate (6.7) with θ in place of θ^ϵ .

Now applying Proposition 1.3 to the approximate equation (6.1) (with $\tilde{\theta}^\epsilon(t) := \theta^\epsilon(t + t_0)$ replacing $\theta^\epsilon(t)$) and Fatou's lemma, we get that for every $\beta \in]1 - \alpha + \sigma, 1[$ and every $T > t_0 + t_1$,

$$\sup_{t \in [t_0 + t_1, T]} \|\theta(t)\|_{\dot{C}^\beta(\mathbb{R}^d)} \leq f(\|\theta_0\|_{L^2}, t_0, \alpha, \beta, \sigma, d, T), \quad (6.14)$$

with t_1 the time introduced above. Hence, the estimate (6.14) yields

$$\begin{aligned} \sup_{t \in [t_0 + t_1, T]} \|u(t)\|_{C^\beta} &\leq C \sup_{t \in [t_0 + t_1, T]} \|\theta(t)\|_{L^2} + C \sup_{t \in [t_0 + t_1, T]} \|\theta(t)\|_{\dot{C}^\beta} \\ &\leq C f(\|\theta_0\|_{L^2}, t_0, \alpha, \beta, \sigma, d, T), \end{aligned}$$

which together with Lemma 2.5 implies the $C_{x,t}^\infty$ -regularity of $\theta(x, t)$ for all $t \in]t_0 + t_1, T]$.

Besides, if $\alpha \in]0, 1[$ and $\sigma = 0$ in the condition (1.9), i.e. $m(y) = C_0|y|^\alpha$, $\forall y \neq 0$, from (2.11), we have that there is no term $-\|\theta^\epsilon\|_{L^2}^2$ in (6.5) and the constant C in the R.H.S. of (6.6), (6.7) and (6.12) can be replaced with the constant 0, which guarantees T in (6.13)-(6.14) can be chosen to be ∞ . Next by choosing $\beta = 1 - \frac{\alpha}{2}$, we see that $\gamma = \frac{\alpha^4}{C}$, and (5.12) just reduces to

$$t_1 \leq \frac{C}{\alpha} \left(\frac{C(1-\alpha)}{\alpha^5} \right)^{\frac{\alpha}{1-\alpha}} \|\theta(t_0)\|_{L^\infty}^{\frac{\alpha}{1-\alpha}}; \quad (6.15)$$

moreover (6.7) becomes

$$\|\theta(t_0)\|_{L^\infty} \leq \left(\frac{C 2^{d/\alpha}}{t_0} \right)^{d/(2\alpha)} \|\theta_0\|_{L^2}, \quad (6.16)$$

which combining with (6.15) leads to (1.26). Thus we finish the proof of Theorem 1.4.

6.2. Proof of Theorem 1.5. In this subsection, let $\theta \in C([0, T^*]; H^s(\mathbb{R}^d)) \cap C^\infty([0, T^*] \times \mathbb{R}^d)$ be the maximal lifespan solution for the drift-diffusion equation (1.1)-(1.2). According to Theorem 7.1, the maximal existence time T^* satisfies $T^* \geq T_1$ with T_1 given by (7.9). Since $\|\theta_0\|_{H^s(\mathbb{R}^d)} \leq R$, we see that

$$T^* \geq T_1 > 1/(2\tilde{C}R), \quad (6.17)$$

with $\tilde{C} = \tilde{C}(d) > 0$ the constant appearing in (7.9).

Next, by setting σ small enough, we show that the eventual regularity time t_1 obtained similarly as obtained in Proposition 1.3 is smaller than the above T_1 . From (5.7), we assume $\xi_0(0) = A_0 \leq \frac{c_0}{2}$

with no loss of generality. Analogous with (5.1)-(5.2), we consider the following family of moduli of continuity that for $\xi_0 \in]\delta, A_0]$,

$$\omega(\xi, \xi_0) = \begin{cases} (1 - \beta)\kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi_0} m(\eta^{-1})d\eta - \gamma m(\xi_0^{-1})(\xi_0 - \delta) + \beta\kappa m(\delta^{-1})\xi, & \text{for } 0 < \xi \leq \delta, \\ \kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi_0} m(\eta^{-1})d\eta - \gamma m(\xi_0^{-1})\xi_0 + \gamma m(\xi_0^{-1})\xi, & \text{for } \delta < \xi \leq \xi_0, \\ \kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi} m(\eta^{-1})d\eta, & \text{for } \xi_0 < \xi \leq c_0, \\ \omega(c_0, \xi_0), & \text{for } \xi > c_0, \end{cases} \quad (6.18)$$

and for $\xi_0 \leq \delta$,

$$\omega(\xi, \xi_0) = \begin{cases} (1 - \beta)\kappa m(\delta^{-1})\delta^{1-\beta}\xi_0^{\beta} + \beta\kappa m(\delta^{-1})\delta^{1-\beta}\xi_0^{\beta-1}\xi, & \text{for } 0 < \xi \leq \xi_0, \\ \kappa m(\delta^{-1})\delta^{1-\beta}\xi^{\beta}, & \text{for } \xi_0 < \xi \leq \delta, \\ \kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{\xi} m(\eta^{-1})d\eta, & \text{for } \delta < \xi \leq c_0, \\ \omega(c_0, \xi_0), & \text{for } \xi > c_0, \end{cases} \quad (6.19)$$

where $\beta \in]\sigma, 1[$, and κ, γ, δ are appropriate positive constants, and $\xi_0 = \xi_0(t)$ is given by (5.5). Recalling that B_0 defined by (4.10) is the bound of $\|\theta(\cdot, t)\|_{L^\infty}$ appearing in Lemma 2.4, and thanks to $\alpha = 1$ and $m(A_0^{-1})A_0^{1-\sigma} \geq m(c_0^{-1})c_0^{1-\sigma} =: \tilde{c}_0$, the condition (5.4) with B_0 in place of $\|\theta_0\|_{L^\infty}$ holds true provided that

$$\frac{(1 - \sigma)\gamma}{\sigma} \tilde{c}_0 (A_0^\sigma - \delta^\sigma) \geq 2B_0, \quad (6.20)$$

which can further be implied by choosing A_0 and δ as

$$A_0 = \left(\frac{\sigma}{(1 - \sigma)\gamma \tilde{c}_0} 4B_0 \right)^{1/\sigma}, \quad \delta = \left(\frac{\sigma}{(1 - \sigma)\gamma \tilde{c}_0} B_0 \right)^{1/\sigma}. \quad (6.21)$$

By using (6.20), we also see that the MOC defined by (6.18)-(6.19) satisfies $\omega(A_0, \xi_0) \geq \omega(A_0, 0+) > 2B_0$ for all $0 < \xi_0 \leq A_0$, thus we only need to justify the criterion

$$- \partial_{\xi_0} \omega(\xi, \xi_0) \dot{\xi}_0(t) + \Omega_{x,e}(\xi, t) \partial_\xi \omega(\xi, \xi_0) + D_{x,e}(\xi, t) + \epsilon \partial_{\xi\xi} \omega(\xi, \xi_0) < 0, \quad (6.22)$$

for all $0 < \xi_0 \leq A_0$, $0 < \xi \leq A_0$, $A_0 \leq \frac{c_0}{2}$, with

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C'_1 \omega(\xi, \xi_0) + C_1 \int_0^{\frac{\xi}{2}} (\omega(\xi + 2\eta, \xi_0) + \omega(\xi - 2\eta, \xi_0) - 2\omega(\xi, \xi_0)) \frac{m(\eta^{-1})}{\eta} d\eta \\ &\quad + C_1 \int_{\frac{\xi}{2}}^{A_0} (\omega(2\eta + \xi, \xi_0) - \omega(2\eta - \xi, \xi_0) - 2\omega(\xi, \xi_0)) \frac{m(\eta^{-1})}{\eta} d\eta, \end{aligned} \quad (6.23)$$

and

$$\Omega_{x,e}(\xi, t) \leq -\frac{C_2}{m(\xi^{-1})} D_{x,e}(\xi, t) + (C_2 + C'_2) \omega(\xi, \xi_0) + C_2 \xi \int_\xi^\infty \frac{\omega(\eta, \xi_0)}{\eta^2} d\eta, \quad (6.24)$$

and $C_1, C_2 > 0$, and $C'_1, C'_2 \geq 0$ obeying the same convection stated after (4.13). By arguing as Proposition 1.3, we indeed can prove this issue as long as that ρ, κ, γ are suitable constants satisfying (5.56) (maybe with slightly different C) and A_0 defined by (6.21) additionally satisfies

$$A_0 \leq \min \left\{ \left(\frac{C_1 \tilde{c}^2 (1 - \beta)(1 - \sigma)}{64 C'_1} \right)^{1/(1-\sigma)}, \frac{c_0}{2} \right\}. \quad (6.25)$$

We will present the different points compared with the proof of Lemma 5.1 in the end of the subsection.

Taking advantage of $m(A_0^{-1})A_0^{1-\sigma} \geq \tilde{c}_0 := m(c_0^{-1})c_0^{1-\sigma}$ again, (5.7) is ensured if we have

$$t_1 \leq \frac{A_0^{1-\sigma}}{(1 - \sigma)\rho \tilde{c}_0} = \frac{1}{(1 - \sigma)\rho \tilde{c}_0} \left(\frac{4\sigma B_0}{(1 - \sigma)\gamma \tilde{c}_0} \right)^{\frac{1-\sigma}{\sigma}}. \quad (6.26)$$

According to (5.56), we find that the conditions on ρ, γ are

$$0 < \rho \leq \frac{(1-\beta)(1-\sigma)}{C}, \quad 0 < \gamma \leq \frac{1}{C} \min \{ (1-\beta)^3(1-\sigma), \beta(1-\beta)^2 \},$$

with $\beta \in]\sigma, 1[$ and $C > 0$ some constant depending on C_1, C_2, C'_2 . Due to that σ is sufficiently small, we choose $\beta = \frac{1}{2}$, and $\sigma \leq \frac{1}{3}$, and ρ, γ to be fixed positive constants (depending only on d). Noticing that for $\alpha = 1$, $\sigma > 0$ small enough, the bound of $\|\theta(\cdot, t)\|_{L^\infty}$ considered in Lemma 2.4 is $\max \{C_d \|\theta_0\|_{L^2}, \|\theta_0\|_{L^\infty}\}$ which does not depend on σ , we infer that (6.25) holds provided that

$$A_0 \leq \min \left\{ \left(\frac{C_1 \tilde{c}^2}{256 C'_1} \right)^2, \frac{c_0}{2} \right\} =: \tilde{A}_0, \quad (6.27)$$

where we have assumed $\frac{C_1 \tilde{c}^2}{256 C'_1} \leq 1$ without loss of generality. Thus by using $\|\theta_0\|_{L^2 \cap L^\infty} \leq R$, $B_0 \leq C_d R$ and $1 - \sigma \geq \frac{2}{3}$, (6.26) becomes

$$t_1 \leq \frac{3}{2\rho \tilde{c}_0} \left(\frac{6C_d}{\gamma \tilde{c}_0} \sigma R \right)^{\frac{1-\sigma}{\sigma}}, \quad (6.28)$$

and (6.25) is guaranteed if the following inequality holds

$$\left(\frac{6C_d}{\gamma \tilde{c}_0} \sigma R \right)^{\frac{1}{\sigma}} \leq \tilde{A}_0. \quad (6.29)$$

Hence, in order to let $t_1 \leq (2\tilde{C}R)^{-1}$ and (6.29) be satisfied, we need

$$\sigma \leq \min \left\{ \frac{\gamma \tilde{c}_0}{6C_d} \left(\frac{\rho \tilde{c}_0}{3\tilde{C}} \right)^{\frac{\sigma}{1-\sigma}} \frac{1}{R^{\frac{1}{1-\sigma}}}, \frac{\gamma \tilde{c}_0 \tilde{A}_0^\sigma}{6C_d} \frac{1}{R} \right\},$$

and by assuming $\frac{\rho \tilde{c}_0}{3\tilde{C}} \leq 1$ and $R \geq 1$ without loss of generality, it suffices to choose σ small enough such that

$$\sigma \leq \min \left\{ \frac{\gamma \tilde{c}_0}{6C_d} \left(\frac{\rho \tilde{c}_0}{3\tilde{C}} \right)^{1/2} \frac{1}{R^{3/2}}, \frac{\tilde{A}_0^{1/3} \gamma \tilde{c}_0}{6C_d} \frac{1}{R}, \frac{1}{3} \right\} := \sigma_1.$$

For such a small number σ , we have $t_1 < T_1$, and thus by virtue of (5.14), we obtain that the solution $\theta(x, t)$ is $\frac{1}{2}$ -Hölder continuous for all $t \in [t_1, T^*]$, which in combination with the blowup criterion (7.2) implies $T^* = \infty$. This concludes Theorem 1.5.

Finally, we state the different points of proving (6.22) in the considered cases, compared to that of Lemma 5.1.

Case 1: $\delta < \xi_0 \leq A_0$, $0 < \xi \leq \delta$.

Since $\partial_\eta \omega(\eta, \xi_0) = 0$ for all $\eta > c_0$, we can prove the estimates analogous to (5.21) and (5.22) with $C_2 + C'_2$ in place of C_2 . For the contribution from the diffusion term, we have (noting that $\alpha = 1$)

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C'_1 \omega(\xi, \xi_0) - 2C_1 \omega(0+, \xi_0) \int_{\frac{\xi}{2}}^{A_0} \frac{m(\eta^{-1})}{\eta} d\eta \\ &\leq C'_1 \omega(\xi, \xi_0) - 2C_1 \omega(0+, \xi_0) \left(\frac{\xi}{2} \right) m \left(\frac{2}{\xi} \right) \int_{\frac{\xi}{2}}^{\xi} \frac{1}{\eta^2} d\eta \\ &\leq C'_1 (\omega(0+, \xi_0) + \beta \kappa m(\delta^{-1}) \delta) - \frac{C_1}{2} \omega(0+, \xi_0) m(\xi^{-1}) \\ &\leq C'_1 \beta \kappa m(\delta^{-1}) \delta - \frac{C_1}{4} \omega(0+, \xi_0) m(\xi^{-1}), \end{aligned} \quad (6.30)$$

where in the last line we used $m(A_0^{-1}) \geq \frac{4C'_1}{C_1}$, or a stronger condition $A_0 \leq \left(\frac{C_1}{4C'_1}\right)^{1/(1-\sigma)}$. Since $\omega(0+, \xi_0) \geq M_{\xi_0, \delta}$ by (5.18), thus if $\xi_0 \leq N\delta$ with $N \in \mathbb{N}$ defined by (5.25), in view of (5.26), we get

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C'_1 \beta \kappa m(\delta^{-1}) \delta - \frac{C_1(1-\beta)\kappa}{4} m(\delta^{-1}) \delta m(\xi^{-1}) - \frac{C_1(1-\sigma)\gamma}{4} m(\xi_0^{-1}) \xi_0 m(\xi^{-1}) \\ &\leq -\frac{C_1(1-\beta)\kappa}{8} m(\delta^{-1}) \delta m(\xi^{-1}) - \frac{C_1(1-\sigma)\gamma}{4} m(\xi_0^{-1}) \xi_0 m(\xi^{-1}), \end{aligned}$$

where in the second line we used $A_0 \leq \left(\frac{C_1(1-\beta)}{8C'_1\beta}\right)^{\frac{1}{1-\sigma}}$; whereas if $\xi_0 \leq N\delta$, by virtue of (5.31), we see that by setting $\gamma \leq \frac{(1-\beta)(1-\sigma)\kappa}{16}$,

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C'_1 \beta \kappa m(\delta^{-1}) \delta - \frac{C_1}{4} ((1-\beta)\kappa m(\delta^{-1}) \delta - \gamma m(\xi_0^{-1}) \xi_0) m(\xi^{-1}) \\ &\leq -\frac{C_1}{4} \left(\frac{(1-\beta)\kappa}{2} - \frac{4\gamma}{1-\sigma} \right) (m(\delta^{-1}))^2 \delta \leq -\frac{C_1(1-\beta)\kappa}{16} (m(\delta^{-1}))^2 \delta, \end{aligned}$$

where we also used $A_0 \leq \left(\frac{C_1(1-\beta)}{8C'_1\beta}\right)^{\frac{1}{1-\sigma}}$. Thus under the conditions (5.29), (5.32) (up to some pure numbers and C_2 replaced by $C_2 + C'_2$), we show that (6.22) holds in this case.

Case 2: $\delta < \xi_0 \leq A_0$, $\delta < \xi \leq \xi_0$.

The different points are quite similar to those stated in Case 1 above, and under the (slightly modified) conditions (5.36) and (5.37), we can show (6.22) in this case.

Case 3: $\delta < \xi_0 \leq A_0$, $\xi_0 < \xi \leq A_0$.

We also obtain (5.38) with C_2 replaced by $C_2 + C'_2$. For $D_{x,e}(\xi, t)$, similarly as (5.39) and (6.30), we have

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C'_1 \omega(\xi, \xi_0) + C_1 (\omega(2\xi, \xi_0) - 2\omega(\xi, \xi_0)) \int_{\frac{\xi}{2}}^{A_0} \frac{m(\eta^{-1})}{\eta} d\eta \\ &\leq C'_1 \omega(\xi, \xi_0) + \frac{C_1}{4} (\omega(2\xi, \xi_0) - 2\omega(\xi, \xi_0)) m(\xi^{-1}), \end{aligned}$$

thus if $\xi \geq \delta \left(\frac{1}{2^{1-\sigma}-1}\right)^{1/\sigma}$, by using (5.41), we get

$$D_{x,e}(\xi, t) \leq C'_1 \omega(\xi, \xi_0) - \frac{C_1 \tilde{c}(1-\sigma)}{8} \omega(\xi, \xi_0) m(\xi^{-1}) \leq -\frac{C_1 \tilde{c}(1-\sigma)}{16} \omega(\xi, \xi_0) m(\xi^{-1}),$$

where in the last inequality we used $A_0 \leq \left(\frac{C_1 \tilde{c}(1-\sigma)}{16C'_1}\right)^{\frac{1}{1-\sigma}}$. If $\xi \leq \delta \left(\frac{1}{2^{1-\sigma}-1}\right)^{1/\sigma}$, by arguing as (5.43) and (6.30), and using (5.44), we find

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C'_1 \omega(\xi, \xi_0) - \frac{\omega(0+, \xi_0)}{2} m(\xi^{-1}) \\ &\leq C'_1 \left(\kappa + \frac{\tilde{c}\gamma}{2(1-\sigma)} \right) m(\delta^{-1}) \delta - \frac{(1-\beta)\kappa}{2} m(\delta^{-1}) \delta m(\xi^{-1}) \\ &\leq -\frac{(1-\beta)\kappa}{4} m(\delta^{-1}) \delta m(\xi^{-1}), \end{aligned}$$

where in the last line we used $\gamma \leq \kappa$ and $A_0 \leq \left(\frac{C_1(1-\beta)(1-\sigma)}{4C'_1}\right)^{\frac{1}{1-\sigma}}$. The remaining proof is similar to that of Case 3 in Section 5, and (6.22) holds in this case under (slightly modified) (5.42) and (5.45).

Case 4: $0 < \xi_0 \leq \delta$, $0 < \xi \leq \xi_0$.

We also have (5.49) with C_2 replaced by $C_2 + C'_2$. In a similar treatment as (5.50) and (6.30), we infer

$$D_{x,e}(x, t) \leq C'_1 \omega(\xi, \xi_0) - \frac{C_1}{2} \omega(0+, \xi_0) m(\xi^{-1}) \leq -\frac{C_1}{4} \omega(0+, \xi_0) m(\xi^{-1}),$$

where in the last inequality we used (5.46) and $A_0 \leq (C_1/(4C'_1))^{\frac{1}{1-\sigma}}$. Thus we can obtain (6.22) in this case under (slightly modified) (5.51).

Case 5: $0 < \xi_0 \leq \delta$, $\xi_0 < \xi \leq \delta$.

We also have (5.52) with C_2 replaced by $C_2 + C'_2$. If $\xi_0 \leq \frac{\xi}{4}$, (5.53) and (5.47) lead to

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C'_1 \omega(\xi, \xi_0) - \frac{C_1 \beta (1-\beta) \kappa}{32} m(\delta^{-1}) \delta^{1-\beta} \xi^\beta m(\xi^{-1}) \\ &\leq -\frac{C_1 \beta (1-\beta) \kappa}{64} (m(\delta^{-1}))^2 \delta^{2-\sigma-\beta} \xi^{\beta-1+\sigma}, \end{aligned}$$

where in the last line we used $A_0 \leq \left(\frac{C_1 \beta (1-\beta)}{64 C'_1}\right)^{\frac{1}{1-\sigma}}$; whereas if $\xi_0 \geq \frac{\xi}{4}$, by arguing as (5.54) and (6.30), we obtain

$$\begin{aligned} D_{x,e}(\xi, t) &\leq C'_1 \omega(\xi, \xi_0) - \frac{C_1}{2} \omega(0+, \xi_0) m(\xi^{-1}) \\ &\leq C'_1 \omega(\xi, \xi_0) - \frac{C_1 (1-\beta)}{8} \omega(\xi, \xi_0) m(\xi^{-1}) \leq -\frac{C_1 (1-\beta) \kappa}{16} (m(\delta^{-1}))^2 \delta^{2-\sigma-\beta} \xi^{\beta-1+\sigma}, \end{aligned}$$

where the last inequality is deduced from using $A_0 \leq \left(\frac{C_1 (1-\beta)}{16 C'_1}\right)^{\frac{1}{1-\sigma}}$. Thus we can similarly obtain (6.22) in this case under (slightly modified) (5.55).

Case 6: $0 < \xi_0 \leq \delta$, $\delta < \xi \leq A_0$.

The proof is almost the same as that of Case 2 in Section 4 by only setting $b_0 = A_0$, and the condition on A_0 is expressed by (4.22) with $\alpha = 1$.

Therefore, gathering the above results concludes (6.22) and thus Theorem 1.5.

6.3. Proof of Theorem 1.7: global regularity of weak solution in the logarithmically supercritical case. Considering the ϵ -regularized equation (6.1), by virtue of Lemma 6.1, there is a uniquely global smooth solution $\theta^\epsilon(x, t)$ to the system (6.1) so that

$$\theta^\epsilon \in C([0, \infty[; H^s(\mathbb{R}^d)) \cap C^\infty([0, \infty[\times \mathbb{R}^d), \quad \text{with } s > d/2 + 1.$$

According to Lemma 2.4, we have the uniform-in- ϵ L^∞ -estimate $\sup_{t \geq 0} \|\theta^\epsilon(t)\|_{L^\infty} \leq B_0$ with B_0 defined by (4.10) in both cases, and the uniform energy estimate $\|\theta^\epsilon(t)\|_{L^2} \leq \|\theta_0\|_{L^2}$, $\forall t \geq 0$ if Case (II) is considered.

According to Proposition 1.3 and Theorem 1.5, the evolution $\theta^\epsilon(x, t)$ uniformly-in- ϵ preserves the MOC $\omega(\xi, \xi_0)$ defined by (6.18)-(6.19) with $\xi_0 = \xi_0(t)$ given by (5.5) and $\rho, \kappa, \gamma > 0$ fixed constants satisfying (5.56), as long as $A_0 = \xi_0(0) > \delta$ simultaneously satisfies $\omega(0+, A_0) > 2B_0$ and

$$A_0 \leq \min \left\{ \left(\frac{C_1 \tilde{c}^2 (1-\beta)(1-\sigma)}{64 C'_1} \right)^{1/(1-\sigma)}, \frac{c_\sigma}{2} \right\}, \quad (6.31)$$

with C_1, C'_1 some constants ($C'_1 = 0$ if Case (I) is considered).

From (5.1), without loss of generality assuming $A_0 \leq c_2^{-1}$ (with c_2 appearing in (1.27)), we see that

$$\begin{aligned}
\omega(0+, A_0) &= (1 - \beta)\kappa m(\delta^{-1})\delta + \gamma \int_{\delta}^{A_0} m(\eta^{-1})d\eta - \gamma m(A_0^{-1})(A_0 - \delta) \\
&\geq \frac{\gamma}{c_2} \int_{\delta}^{A_0} \frac{1}{\eta(\log \eta^{-1})^{\mu}} d\eta - \gamma \\
&\geq \frac{\gamma}{c_2} \int_{\frac{1}{A_0}}^{\frac{1}{\delta}} \frac{1}{\eta(\log \eta)^{\mu}} d\eta - \gamma \\
&\geq \begin{cases} \frac{\gamma}{c_2(1-\mu)} \left(\left(\log \frac{1}{\delta} \right)^{1-\mu} - \left(\log \frac{1}{A_0} \right)^{1-\mu} \right) - \gamma, & \text{if } \mu \in [0, 1[, \\ \frac{\gamma}{c_2} \left(\log \log \frac{1}{\delta} - \log \log \frac{1}{A_0} \right) - \gamma, & \text{if } \mu = 1. \end{cases}
\end{aligned} \tag{6.32}$$

In order to let $\omega(0+, A_0) > 2B_0$, if $\mu \in [0, 1[$, we need

$$\log \frac{1}{\delta} > \left[\left(\log \frac{1}{A_0} \right)^{1-\mu} + \frac{c_2(1-\mu)}{\gamma} (2B_0 + \gamma) \right]^{\frac{1}{1-\mu}},$$

and from the inequality $(a + b)^{\frac{1}{1-\mu}} \leq C_{\mu}(a^{\frac{1}{1-\mu}} + b^{\frac{1}{1-\mu}})$ for $a, b > 0$, it suffices to choose δ as

$$\delta = A_0^{C_{\mu}} \exp \left(-C_{\mu} \left(\frac{c_2(1-\mu)}{\gamma} (3B_0 + \gamma) \right)^{1/(1-\mu)} \right); \tag{6.33}$$

whereas if $\mu = 1$, it suffices to set δ as

$$\log \log \frac{1}{\delta} = \log \log \frac{1}{A_0} + \frac{c_2}{\gamma} (3B_0 + \gamma),$$

that is,

$$\delta = A_0^{\exp \left(\frac{c_2}{\gamma} (3B_0 + \gamma) \right)}. \tag{6.34}$$

From (5.56), the conditions on $\rho, \kappa, \gamma > 0$ are

$$\rho \leq \frac{1}{C}(1 - \beta)(1 - \sigma), \quad \kappa \leq \frac{1}{C}(1 - \beta)^2, \quad \gamma \leq \frac{1}{C} \min \{ \beta(1 - \beta)^2, (1 - \beta)^3(1 - \sigma) \},$$

with $C > 0$ the suitable constant depending only on d and c_1 . Since we may let σ, β small enough, we assume $\sigma \in]0, \frac{1}{4}[$ and $\beta \in]\sigma, \frac{1}{2}[$, and thus we can set $\rho = \frac{1}{C'}$, $\kappa = \frac{1}{C'}$ and $\gamma = \frac{\beta}{C'}$ with some pure number $C' > 0$. Thanks to (5.7), (1.27) and (5.10), we find that the eventual regularity time t_1 satisfies

$$t_1 \leq \frac{1}{(1 - \sigma)\rho m(A_0^{-1})} \leq 2C' A_0 (\log A_0^{-1})^{\mu} \leq 2C' C_0^{\mu} A_0^{1-\frac{\mu}{2}} \leq 2C' C_0 A_0^{\frac{1}{2}}, \tag{6.35}$$

and for every $\beta \in]\sigma, 1[$,

$$\begin{aligned}
&\sup_{t \in [t_1, \infty[} \|\theta^{\epsilon}(t)\|_{\dot{C}^{\beta}(\mathbb{R}^d)} \leq \kappa m(\delta^{-1})\delta^{1-\beta} \leq c_2 \kappa \delta^{-\beta} \\
&\leq \begin{cases} \frac{c_2}{C'} A_0^{-C_{\mu}\beta} \exp \left(\beta C_{\mu} \left(\frac{3C' c_2(1-\mu)B_0}{\beta} + c_2(1-\mu) \right)^{1/(1-\mu)} \right), & \text{if } \mu \in [0, 1[, \\ \frac{c_2}{C'} (A_0^{-1})^{\beta \exp \left(\frac{3C' c_2 B_0}{\beta} + c_2 \right)}, & \text{if } \mu = 1, \end{cases}
\end{aligned} \tag{6.36}$$

where the condition on A_0 is as follows (from (6.31) and (6.27))

$$0 < A_0 \leq \min \left\{ \left(\frac{C_1 \tilde{c}^2}{256 C_1'} \right)^2, \frac{c_{\sigma}}{2}, c_2^{-1} \right\}.$$

Now for any $t_* > 0$, by virtue of (6.35), we also need A_0 satisfies that $2C'C_0A_0^{\frac{1}{2}} \leq \frac{t_*}{2}$, i.e. $A_0 \leq \left(\frac{t_*}{4C'C_0}\right)^2$, thus for each $\sigma \in]0, \frac{1}{4}[$ and $\beta \in]\sigma, \frac{1}{2}[$, we can choose A_0 to be

$$A_0 = \min \left\{ \left(\frac{C_1 \tilde{c}^2}{256C'_1} \right)^2, \frac{c_\sigma}{2}, c_2^{-1}, \left(\frac{t_*}{4C'C_0} \right)^2 \right\}, \quad (6.37)$$

so that the uniform-in- ϵ Hölder estimate (6.36) holds true. According to Lemma 2.5 and the Calderón-Zygmund theorem, we can further show the uniform bound of $\sup_{t \in [t_*, \infty[} \|\theta^\epsilon(t)\|_{C^{k, \nu}}$ for any $k \in \mathbb{N}$ and $\nu \in [0, 1[$. Since $t_* > 0$ is arbitrarily given, such a uniform bound ensures that we can pass to the limit $\epsilon \rightarrow 0$ in the approximate equation (6.1) to obtain a weak solution θ of the original equation (1.1)-(1.2), which satisfies $\theta \in L^\infty([t_*, \infty[; C^\infty(\mathbb{R}^d))$ and also $\theta \in C^\infty([t_*, \infty[\times \mathbb{R}^d)$. Thus we conclude Theorem 1.7.

7. APPENDIX: LOCAL WELL-POSEDNESS RESULT

Proposition 7.1. *Assume that $\theta_0 \in H^s(\mathbb{R}^d)$, $s > 1 + \frac{d}{2}$, and either Case (I) (cf. (1.19)) or Case (II) (cf. (1.20)) is considered. Then there exists a time $T > 0$ depending only on $\|\theta_0\|_{H^s}$ and dimension d such that the drift-diffusion equation (1.1)-(1.2) admits a uniquely local smooth solution $\theta(x, t)$, which satisfies*

$$\theta \in C([0, T[; H^s(\mathbb{R}^d)) \cap L^2([0, T[; H^{s+\frac{\alpha-\sigma}{2}}(\mathbb{R}^d)) \cap C^\infty(]0, T[\times \mathbb{R}^d). \quad (7.1)$$

Moreover, let T^* be the maximal existence time of the solution satisfying (7.1), then we necessarily get that

$$\text{if } T^* < \infty \quad \Rightarrow \quad \|\theta\|_{L^\infty([0, T^*]; \dot{C}^\beta(\mathbb{R}^d))} = \infty, \quad \forall \beta \in]1 - \alpha + \sigma, 1[. \quad (7.2)$$

Before proving this local result, we introduce the definition of Besov spaces. First one can choose two nonnegative radial functions $\chi, \varphi \in C_c^\infty(\mathbb{R}^d)$ (cf. [2]) be supported respectively in the ball $\{\zeta \in \mathbb{R}^d : |\zeta| \leq \frac{4}{3}\}$ and the shell $\{\zeta \in \mathbb{R}^d : \frac{3}{4} \leq |\zeta| \leq \frac{8}{3}\}$ such that $\chi(\zeta) + \sum_{q \geq 0} \varphi(2^{-q}\zeta) = 1$, $\forall \zeta \in \mathbb{R}^d$. For all $f \in \mathcal{S}'(\mathbb{R}^d)$, we define the nonhomogeneous Littlewood-Paley operators as

$$\Delta_{-1}f := \chi(D)f; \quad \Delta_q f := \varphi(2^{-q}D)f, \quad \forall q \in \mathbb{N}.$$

Then for $(p, r) \in [1, \infty]^2$, $s \in \mathbb{R}$, the nonhomogeneous Besov space is denoted by

$$B_{p,r}^s(\mathbb{R}^d) = B_{p,r}^s := \left\{ f \in \mathcal{S}'(\mathbb{R}^d); \|f\|_{B_{p,r}^s} := \|\{2^{qs}\|\Delta_q f\|_{L^p}\}_{q \geq -1}\|_{\ell^r} < \infty \right\}.$$

We point out that $B_{2,2}^s = H^s$ for every $s \in \mathbb{R}$, and $B_{\infty,\infty}^s = C^s$ for every $s \in \mathbb{R} \setminus \mathbb{Z}$.

Proof of Theorem 7.1. We first are concerned with the key *a priori* estimates, then we sketch the main process of the proof.

Step 1: *a priori* estimates.

We assume θ is already a smooth solution of (1.1). For every $q \in \mathbb{N}$, by applying Δ_q to the equation (1.1), we get

$$\partial_t \Delta_q \theta + u \cdot \nabla \Delta_q \theta + \mathcal{L}(\Delta_q \theta) = F_q,$$

with $F_q = u \cdot \nabla \Delta_q \theta - \Delta_q(u \cdot \nabla \theta)$. Multiplying both sides of the above equation with $\Delta_q \theta$ ($q \in \mathbb{N}$) and integrating on the x -variable over \mathbb{R}^d lead to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_q \theta(t)\|_{L^2}^2 + \int_{\mathbb{R}^d} \mathcal{L}(\Delta_q \theta)(x, t) \Delta_q \theta(x, t) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^d} (\operatorname{div} u)(x, t) |\Delta_q \theta(x, t)|^2 dx + \int_{\mathbb{R}^d} F_q(x, t) \Delta_q \theta(x, t) dx. \end{aligned} \quad (7.3)$$

By virtue of (1.13) and the fact $K \geq 0$, we see that the symbol of \mathcal{L} , denoted by $A(\zeta)$, is nonnegative, thus the diffusion term satisfies

$$\int_{\mathbb{R}^d} \mathcal{L}(\Delta_q \theta)(x) \Delta_q \theta(x) dx = \int_{\mathbb{R}^d} A(\zeta) |\widehat{\Delta_q \theta}|^2(\zeta) d\zeta \geq 0.$$

On the other hand, according to (2.10) which is on the lower bound of $A(\zeta)$, we also have that for all $q \geq Q_0$ with Q_0 a suitably large number depending on α, σ, d ,

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{L}(\Delta_q \theta)(x) \Delta_q \theta(x) dx &= \int_{\mathbb{R}^d} A(\zeta) |\widehat{\Delta_q \theta}|^2(\zeta) d\zeta \geq C^{-1} 2^{q(\alpha-\sigma)} \|\Delta_q \theta\|_{L^2}^2 - C \|\Delta_q \theta\|_{L^2}^2 \\ &\geq \frac{1}{2C} 2^{q(\alpha-\sigma)} \|\Delta_q \theta\|_{L^2}^2. \end{aligned}$$

The first integral on the R.H.S. of (7.3) follows directly from the Hölder inequality

$$\int_{\mathbb{R}^d} (\operatorname{div} u)(x, t) |\Delta_q \theta(x, t)|^2 dx \leq \|\operatorname{div} u(t)\|_{L^\infty} \|\Delta_q \theta(t)\|_{L^2}^2.$$

For the second integral on the right-hand-side of (7.3), by using the classical commutator estimate (cf. [2, Theorem 3.14]) that for every $s > 0$, $\|F_q\|_{L^2} \leq C c_q 2^{-qs} (\|\nabla u\|_{L^\infty} + \|\nabla \theta\|_{L^\infty}) \|\theta\|_{H^s}$ with $\|c_q\|_{\ell^2} = 1$, we get

$$\begin{aligned} \int_{\mathbb{R}^d} F_q(x, t) \Delta_q \theta(x, t) dx &\leq \|F_q(t)\|_{L^2} \|\Delta_q \theta(t)\|_{L^2} \\ &\leq C c_q 2^{-qs} (\|\nabla u(t)\|_{L^\infty} + \|\nabla \theta(t)\|_{L^\infty}) \|\theta(t)\|_{H^s} \|\Delta_q \theta(t)\|_{L^2}. \end{aligned}$$

Gathering the above estimates yields

$$\begin{aligned} &\frac{d}{dt} \|\Delta_q \theta(t)\|_{L^2}^2 + \frac{1}{C} 1_{\{q \geq Q_0\}} 2^{q(\alpha-\sigma)} \|\Delta_q \theta(t)\|_{L^2}^2 \\ &\leq \|\operatorname{div} u\|_{L^\infty} \|\Delta_q \theta(t)\|_{L^2}^2 + C c_q 2^{-qs} \|\Delta_q \theta(t)\|_{L^2} (\|\nabla u\|_{L^\infty} + \|\nabla \theta\|_{L^\infty}) \|\theta\|_{H^s}, \end{aligned}$$

with $1_{\{q \geq Q_0\}}$ the standard indicator function. By multiplying both sides of the above inequality with 2^{2qs} and summing over $q \in \mathbb{N}$, and using the discrete Hölder inequality, we obtain

$$\begin{aligned} &\frac{d}{dt} \left(\sum_{q \in \mathbb{N}} 2^{2qs} \|\Delta_q \theta(t)\|_{L^2}^2 \right) + \frac{1}{C} \left(\sum_{q \in \mathbb{N}, q \geq Q_0} 2^{2q(s+\frac{\alpha-\sigma}{2})} \|\Delta_q \theta(t)\|_{L^2}^2 \right) \\ &\leq \|\operatorname{div} u\|_{L^\infty} \left(\sum_{q \in \mathbb{N}} 2^{2qs} \|\Delta_q \theta(t)\|_{L^2}^2 \right) + C (\|\nabla u\|_{L^\infty} + \|\nabla \theta\|_{L^\infty}) \|\theta\|_{H^s} \left(\sum_{q \in \mathbb{N}} 2^{2qs} \|\Delta_q \theta(t)\|_{L^2}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (7.4)$$

Now we consider the L^2 energy estimate. In a similar way as obtaining (6.2), we see that

$$\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{L^2}^2 + \int_{\mathbb{R}^d} \mathcal{L}(\theta)(x, t) \theta(x, t) dx = \frac{1}{2} \int_{\mathbb{R}^d} \operatorname{div} u(x, t) |\theta(x, t)|^2 dx, \quad (7.5)$$

which in combination with (6.3) leads to

$$\frac{d}{dt} \|\theta(t)\|_{L^2}^2 \leq \|\operatorname{div} u(t)\|_{L^\infty} \|\theta(t)\|_{L^2}^2. \quad (7.6)$$

Due to the equivalence of norms $\sum_{q \in \mathbb{N}} 2^{2qs} \|\Delta_q \theta(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \approx \|\theta(t)\|_{H^s}^2$, and by gathering (7.4) and (7.6), we deduce

$$\frac{d}{dt} \|\theta(t)\|_{H^s}^2 + \frac{1}{C} \left(\sum_{q \in \mathbb{N}, q \geq Q_0} 2^{2q(s+\frac{\alpha-\sigma}{2})} \|\Delta_q \theta(t)\|_{L^2}^2 \right) \leq C (\|\nabla u(t)\|_{L^\infty} + \|\nabla \theta(t)\|_{L^\infty}) \|\theta(t)\|_{H^s}^2. \quad (7.7)$$

In view of the Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow C^1(\mathbb{R}^d)$ for $s > 1 + \frac{d}{2}$ and the Calderón-Zygmund theorem, we infer that

$$\frac{d}{dt} \|\theta(t)\|_{H^s}^2 + \frac{1}{C} \left(\sum_{q \in \mathbb{N}, q \geq Q_0} 2^{2q(s + \frac{\alpha - \sigma}{2})} \|\Delta_q \theta(t)\|_{L^2}^2 \right) \leq C_1 \|\theta(t)\|_{H^s}^3. \quad (7.8)$$

Denoting by

$$Z(t)^2 := \|\theta(t)\|_{H^s}^2 + \frac{1}{C} \int_0^t \left(\sum_{q \in \mathbb{N}, q \geq Q_0} 2^{2q(s + \frac{\alpha - \sigma}{2})} \|\Delta_q \theta(\tau)\|_{L^2}^2 \right) d\tau,$$

the above inequality implies that $\frac{d}{dt} Z(t) \leq \tilde{C} Z(t)^2$. Through a direct computation, for all t such that

$$t < T_1 := \frac{1}{\tilde{C} \|\theta_0\|_{H^s}}, \quad (7.9)$$

we get

$$Z(t) \leq \frac{\|\theta_0\|_{H^s}}{1 - \tilde{C} \|\theta_0\|_{H^s} t}. \quad (7.10)$$

Thanks to the dyadic decomposition, we also obtain

$$\int_0^t \|\theta(\tau)\|_{H^{s + \frac{\alpha - \sigma}{2}}}^2 d\tau \leq C 2^{Q_0(\alpha - \sigma)} \left(\sup_{\tau \in [0, t]} Z(\tau)^2 \right) t + C Z(t)^2, \quad (7.11)$$

which is bounded for all $t < T_1$.

Step 2: local existence and blowup criterion.

Now we regularize the equation (1.1) to obtain

$$\partial_t \theta^N + J_N (J_N u^N \cdot \nabla J_N \theta^N) + J_N^2 \mathcal{L}(\theta^N) = 0, \quad u^N = \mathcal{P}(\theta^N), \quad \theta^N|_{t=0} = J_N \theta_0, \quad (7.12)$$

where $J_N : L^2 \rightarrow J_N L^2$ for $N \in \mathbb{N}$ is the Friedrich projection operator such that $\widehat{J_N f}(\zeta) = 1_{\{|\zeta| \leq N\}}(\zeta) \widehat{f}(\zeta)$. By the Cauchy-Lipschitz theorem, for every $N \in \mathbb{N}$ there exists a unique solution $\theta^N = J_N \theta^N \in C^1([0, T_N^*]; H^\infty(\mathbb{R}^d))$ to the regularized system (7.12), where $T_N^* > 0$ is the maximal existence time such that $\sup_{t \in [0, T_N^*]} \|\theta^N(t)\|_{L^2} = \infty$. In a similar way as obtaining (7.10)-(7.11), and by using the fact $\|J_N \theta_0\|_{H^s} \leq \|\theta_0\|_{H^s}$ for all $N \in \mathbb{N}$, we get that for every $t < T_1 = \frac{1}{\tilde{C} \|\theta_0\|_{H^s}} \leq T_N^*$,

$$\|\theta^N(t)\|_{H^s}^2 + \int_0^t \|\theta(\tau)\|_{H^{s + \frac{\alpha - \sigma}{2}}}^2 d\tau \leq \left(C 2^{Q_0(\alpha - \sigma)} t + C \right) \frac{\|\theta_0\|_{H^s}^2}{(1 - \tilde{C} \|\theta_0\|_{H^s} t)^2}, \quad (7.13)$$

where $C = C(\alpha, \sigma, d)$ is a positive constant independent of N . Based on the uniform-in- N estimate (7.13), and by arguing as the corresponding part of [31, Proposition 4.1], we can show that θ^N is a Cauchy sequence in $C([0, T_1]; L^2(\mathbb{R}^d))$, which implies that θ^N strongly converges to some function $\theta \in C([0, T_1]; L^2(\mathbb{R}^d))$. By a classical deduction, we can prove that θ belonging to $L^\infty([0, T_1]; H^s(\mathbb{R}^d)) \cap L^2([0, T_1]; H^{s + (\alpha - \sigma)/2}(\mathbb{R}^d))$ is indeed a classical solution of (1.1), which is the limiting equation of (7.12). The uniqueness issue in the L^2 -framework, the fact $\theta \in C([0, T_1]; H^s(\mathbb{R}^d))$ and the smoothing effect that $t^\mu \theta \in L_T^\infty H^{s + (\alpha - \sigma)\mu}$ for all $\mu \geq 0$ and $T \in]0, T_1[$ can be similarly treated as [31], and we omit the proof.

Now let T^* be the maximal existence time such that $\theta \in C([0, T^*]; H^s) \cap C^\infty([0, T^*] \times \mathbb{R}^d)$, then we first get the natural blowup criterion that if $T^* < \infty$, we have $\sup_{t \in [0, T^*]} \|\theta(t)\|_{H^s} = \infty$. Moreover, we obtain the following criterion that

$$\text{if } T^* < \infty \quad \Rightarrow \quad \int_0^{T^*} \|\nabla \theta(t)\|_{L^\infty} dt = \infty. \quad (7.14)$$

In order to show this criterion, we rely on the logarithmic estimate (e.g. cf. [29, Lemma 2.3])

$$\|\nabla u\|_{L^\infty} \leq C + C\|\nabla \theta\|_{L^\infty} \log(e + \|\theta\|_{H^s}). \quad (7.15)$$

Plugging (7.15) into (7.7), and integrating on the time variable yields

$$\|\theta(t)\|_{H^s} \leq (e + \|\theta_0\|_{H^s}) \exp \exp \left\{ Ct + \int_0^t \|\nabla \theta(\tau)\|_{L^\infty} d\tau \right\},$$

which directly implies the criterion (7.14). For the more refined criterion (7.2), it is in fact a consequence of the regularity criterion shown in Lemma 2.5. Indeed, for every $T \in]0, T^*[$ and under the condition that $\theta \in L^\infty([0, T]; \dot{C}^\beta)$, $\beta \in]1 - \alpha + \sigma, 1[$, we have $u \in L^\infty([0, T]; \dot{C}^\beta)$, $\beta \in]1 - \alpha + \sigma, 1[$ from the Calderón-Zygmund theorem, then according to the proof of Lemma 2.5, we further deduce that $\theta \in L^\infty([0, T]; C^{1, \gamma})$, $\gamma > 0$, which clearly guarantees $\sup_{t \in [0, T^*[} \|\nabla \theta\|_{L^\infty} < \infty$ and leads to $T^* = \infty$. \square

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